Geometry of String Theory Solitons

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Abstract

In recent years there has been a dramatic progress in the understanding of the non-perturbative structure of various physical theories. In particular string theory has been vastly developed during these years, where a lot of duality conjectures between the different string theories have arisen. The introductory text of this thesis is an attempt to describe this development in short and to make a brief overview of the subject. Special focus is put on solitonic solutions in various field theories, which is the corner stone for these duality conjectures. The introduction of supersymmetry is also essential for the understanding of duality by its natural way of handling BPS-states through the algebra. In string theory, which is not only a supersymmetric theory but also includes gravity, these studies are put together through the discovery of various p-brane solutions to the background field equations. The geometrical structure of these solutions is studied in some of the papers in this thesis. In a generalization to the treatment of p-branes as solutions which break the local vacuum symmetry, the theory of almost product structures (APS-theory) has arisen as the natural candidate to the study of the intricate geometry of these solutions. The last two papers deal with this ansatz where it is also seen that APS-theory is the most natural way of treating all kinds of splitting of manifolds including fibrations, Yang–Mills theory and Kaluza–Klein theory.

This thesis consists of an introductory text and the following five appended research papers, henceforth referred to as Papers I–V:

I. M. Cederwall and M. Holm,
Monopole and dyon spectra in N=2 SYM with higher rank gauge groups,
(hep-th/9603134).

II. T. Adawi, M. Cederwall, U. Gran, M. Holm and B.E.W. Nilsson,
Superembeddings, non-linear supersymmetry and 5-branes,
Int. J. Mod. Phys. A, Vol 13, No. **27** (1998) 4691 (hep-th/9711203).

III. M. Cederwall, U. Gran, M. Holm and B.E.W Nilsson,
Finite tensor deformations of supergravity solitons,
JHEP 02(1999)003, (hep-th/9812144).

IV. M. Holm,
New insights in brane and Kaluza–Klein theory through almost product structures,
Submitted for publication, (hep-th/9812168).

V. M. Holm and N. Sandström,
Curvature relations in almost product manifolds,
Submitted for publication, (hep-th/9904099).

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Göteborg, Aug 1999
Magnus Holm

To my grandmother, Britt Arvidsson.

Conventions

There are always a lot of different conventions used in the physical literature and as this thesis covers a big area it is, of course, difficult to find a convention that covers it all. But for this thesis to be at all readable I had to choose and stick to one of the many possibilities out there.

Space-Time signature $(-,+,+,\ldots)$

Index notation

Target-SuperSpace coordinates	$z^{\underline{M}} = (x^{\underline{m}}, \theta^{\underline{\mu}})$
TSS coordinate indices	$\underline{M},\ \underline{N},\ \underline{P},\ \underline{Q}$
TSS non-coordinate indices	$\underline{A}, \underline{B},\ \underline{C},\ \underline{D}$
TSS oriented non-coord. ind.	$\bar{A}, \bar{B},\ \bar{C},\ \bar{D}$
TS bosonic coordinate indices	$\underline{m},\ \underline{n},\ \underline{p},\ \underline{q}$
TS fermionic coord. indices	$\underline{\mu},\ \underline{\nu},\ \underline{\rho},\ \underline{\sigma}$
TS bosonic non-coord. indices	$\underline{a}, \underline{b},\ \underline{c},\ \underline{d}$
TS fermionic non-coord. indices	$\underline{\alpha}, \underline{\beta},\ \underline{\gamma},\ \underline{\delta}$
TS bos. orient. non-coord. ind.	$\bar{a}, \bar{b},\ \bar{c},\ \bar{d}$
TS ferm. orient. non-coord. ind.	$\bar{\alpha}, \bar{\beta},\ \bar{\gamma},\ \bar{\delta}$
World-SuperSheet coordinates	$z^M = (x^m, \theta^\mu)$
WSS coordinate indices	$M,\ N,\ P,\ Q$
WSS non-coordinate indices	$A, B,\ C,\ D$
WS bosonic coordinate indices	$m,\ n,\ p,\ q$
WS fermionic coord. indices	$\mu,\ \nu,\ \rho,\ \sigma$
WS bosonic non-coord. indices	$a,\ b,\ c,\ d$
WS fermionic non-coord. indices	$\alpha, \beta,\ \gamma,\ \delta$
Normal-Space non-coord. indices	$A',\ B',\ C',\ D'$
NS bosonic non-coord. indices	$a',\ b',\ c',\ d'$
NS fermionic non-coord. indices	$\alpha', \beta',\ \gamma',\ \delta'$
Gauge indices	$i,\ j,\ k,\ l$

Contents

1

Introduction

The progress of theoretical physics the last decades can adequately be expressed by the word symmetry. There is a large number of new symmetries that have arisen in physical theories under this period. Non-abelian gauge symmetry, supersymmetry and duality are perhaps those of greatest importance lately. This introductory text is an attempt to shed some light on these subjects without the need of too much technicality.

In 1954 Yang and Mills introduced a non-abelian generalization to ordinary electro-magnetism where the internal symmetry could be an arbitrary gauge group. Ideas of this kind arose through Oskar Klein's work two decades earlier but then in the form of what today is called Kaluza–Klein theory. Modern Yang–Mills theory is described through the concept of principal bundles, where the gauge group is called the internal symmetry group of the principal bundle and the space-time manifold its base manifold. From bundle analysis it is clear that the gauge theory could equivalently be described in either the total space of the principal bundle or as a twisting of the gauge group along directions only in the base manifold. While ordinary Yang–Mills theory is firmly based on the latter alternative, Kaluza–Klein theory is an example of the former. In Kaluza–Klein theory the dynamics of the gauge field arises through ordinary gravity in the total space. In chapter 2 we will compare these points of view through yet another ansatz, namely that of almost product structures. Almost product structures is seen to be the most general approach to a splitting of a manifold by which principal bundles and Kaluza–Klein theory are only special cases. It is only through this APS ansatz that the true geometrical features of the manifold can be explored. Through this theory, dubbed APS-theory, these two are put on an equal footing and the geometry of such things as the coupling constant in Yang–Mills theory and the dilaton in Kaluza–Klein theory can be explored.

In ordinary field theory we are used to the concept of sources coupled to various fields. These sources are classically built upon conserved currents which are derived from some symmetry through Noether's theorem. By integrating this current over the spatial part of the manifold one obtains the total charge of the source. While studying the free field solutions in ordinary four dimensional field theory one sees that these sources usually can be taken to be point particles carrying the charge in question. These source particles are referred to as fundamental electrically charged particles. The coupling between the source particles and the fields is tunable through a coupling constant. In quantum field theory this coupling constant is used as an expansion parameter in a perturbation series counting the number of loops in certain Feynman diagrams. An interesting feature arising in field theories with non-linear self-interactions are classical solutions which are purely non-perturbative in the sense that they can not be seen in a perturbation expansion. These solutions have the property of being lumps of energy localized at some point in space. They prove to have similar dynamical properties as ordinary particles but are described through a topological current instead of the Noether current for fundamental particles. These objects, called solitons, thus carry a topological charge which in the case of Yang–Mills theory is called magnetic. From Yang–Mills theory the topological origin of these charges is best understood through the winding of the gauge group over the base manifold. In chapter 3 the concept of solitons is explored by studying some general examples including magnetic monopoles in the Georgi–Glashow model. An interesting feature that arises here is the Bogomol'nyi bound by which the mass of the soliton is bounded from below. Solutions satisfying this bound are called BPS saturated and these play an important role in modern theories including string theory. For BPS states the mass is exactly given by the charge of the state.

Maybe the most interesting feature of modern physics is the concept of duality, which finds its origin through these solitonic solutions of ordinary field theory. Duality imposes a symmetry of the theory under the exchange of fundamental electrically charged particles with solitonic magnetically charged objects. As a basic property of the solitons is that their mass is inversely proportional to a coupling constant, the duality symmetry requires that at the same time the coupling constant is replaced by its inverse. This leads to a strong-weak relation through the duality transformation. What is weakly coupled in one theory is strongly coupled in its dual. This opens for the possibility of going beyond perturbation theory in order to understand strongly coupled theories. In the latter part of chapter 3 some toy models for duality are explored including duality in ordinary electro-magnetic field theory.

All these features discussed are given a more natural explanation through the introduction of supersymmetry as a symmetry between the bosonic sector and the fermionic sector of a theory. Here representation theory forces the magnetically charged solitons to be BPS states. The explanation for this is quite simple. In a Yang–Mills–Higgs system the breaking of the gauge symmetry through a vacuum solution makes the W bosons and the Higgs to acquire masses. In representation theory of supersymmetry there is a distinction between massive and massless representations. Massless representations have only half the degrees of freedom of massive representations. Now there are massive representations with central charges which reduces their degrees of freedom to one half when the mass equals the central charge - these are the BPS states. So by requiring that the degrees of freedom are unchanged by the symmetry breaking the solutions are forced by supersymmetry to be BPS states. This is discussed in chapter 4 where the supersymmetric extensions to ordinary Yang–Mills theory is looked at from a duality perspective.

For a theory to be realistic in explaining all kinds of phenomena gravity must be included. The problem with gravity, explained through Einstein's general relativity, is that it is not renormalizable and therefore not unifiable with quantum field theory. As divergences in quantum field theory arise with different signs for bosons and fermions supersymmetric field theories have in general better quantum properties. In chapter 4, extensions of general relativity to supergravity theories are discussed and some background theories of suitable properties for the upcoming orientation towards string theory are presented. It should be stressed, though, that ordinary supergravity is not a consistent theory for quantum gravity by its own. Although the divergences are damped by supersymmetry they are not completely removed.

String theory has today risen from an inspired, but incorrect, attempt in describing the strong force to a potential candidate for a fundamental theory describing all physical interactions. There are basically two genres of string theory, namely original bosonic string theory and the supersymmetrically extended version known as superstring theory. The bosonic string theory is plagued with tachyons in the spectrum and is thus seemingly unphysical. Nevertheless many features of superstring theory appears already in the bosonic version and it can thus be instructive to look at it as a toy model.

When it comes to string theory one usually talks about the first and second revolution. The first revolution appeared in 1984-1985 when the five quantum mechanically consistent string theories were derived and shown to be totally anomaly free. These included the type IIA, type IIB, type I, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$. The massless spectrum of these theories are derived in chapter 5 where some perturbative features are exploited. The highlight of this chapter is the conclusion that quantizing the string in the various consistent backgrounds brings the background fields on shell. That is the background field equations are obtained order by order in the string parameter α' when performing the β-functional calculation of the string coupled to the background fields. Strikingly, the low energy versions fits precisely into the supergravity picture exploited in the previous chapter.

Chapter 6 will mostly be dedicated to the second string revolution which took place in the mid 90's. Here previous evidence of duality in various field theories was lifted into the string theory arena. After a couple of years it was clear to the physical society that the different string theories were merely different sides of the same coin. In this research program the $D = 11$ supergravity theory, which were formerly rejected, was again brought to light, now as the low energy limit of a much larger theory called M-theory. M-theory came to be the name of the theory of which the different string theories are merely some perturbative domains. As all these theories only are known in the low energy limits, probes for all duality conjectures regarding them would preferably be some topologically stable solutions with quantum mechanically stable masses so they could be tracked through different regimes of the theories. These are the p-branes which are BPS-saturated solutions to the background field equations. In chapter 6 the various types of p-branes appearing in the theories are exploited through the so called brane-scan. Interesting features of these solutions are that they are extreme black hole solutions charged under some anti-symmetric tensor field. "Extreme" means that they do not emit any radiation as they have zero temperature and are thus stable quantum mechanically. Again light upon these solutions are brought to us directly from the supersymmetry algebra in which the central charges contain the information of the present anti-symmetric tensor fields and thus the possible p-branes coupled to them.

In some of these p-brane solutions there are additional vector fields living on the brane. These are called D-branes which is short for Dirichlet-branes, a name that arose through the feature of having strings ending on them. An interesting feature concerning these D-brane solutions is that there is a possible correspondence between the gauge theory describing the dynamics of these vector fields and the string theory containing gravity in the bulk surrounded by the D-brane. This is an example of an holographic theory were the physical degrees of freedom equivalently can be described by a bulk theory or by a theory living on the boundary. In the last section of chapter 6 a conjecture due to Maldacena is very briefly presented which possesses precisely these features.

In paper I we derive parts of the monopole and dyon spectra of $N = 2$ super–Yang–Mills theory coupled to matter multiplets. We explicitly derive the moduli space of the $(1, 1)$ monopole of the $SU(3)$ gauge theory which turns out to be a Taub–Nut space times the original moduli space of the $(1, 0)$ monopole. From this paper we saw that there is no naive strong-weak duality in the general theory with higher rank gauge groups.

In paper II we look at the five-brane in seven dimensions from a super-embedding approach. Here we see that the seven-dimensional case differ a bit from the eleven-dimensional in that it does not put the background theory on-shell. To do this an additional constraint on the torsion must be imposed. Through the analysis in this paper we are also able to make a direct connection between the super-embedding approach and the approach of non-linear realization.

In paper III we derive new p-brane solutions with additional excited world-volume tensor fields. The M5-brane and the D3-brane with the world-volume tensor field and vector field respectively excited are derived. Some interesting geometrical features of these solutions are discussed.

In paper IV the geometry of the p-brane solutions together with the theory of fibrations, Yang–Mills theory and Kaluza–Klein theory are all put on equal footing through the theory of almost product structure (APS-theory). From this perspective, the theories can be given their true geometrical and topological features.

In paper V we continue the research of paper IV by deriving all the curvature relations involved in an arbitrary APS-theory. Furthermore are the conformal properties of these curvature components are derived.

2

The theory of almost product structures

In 1954 Yang and Mills generalized the theory of electromagnetism when they introduced the so called Yang–Mills theory. In their approach to describe the weak interactions they introduced a $SU(2)$ gauge field which is the simplest example of a *non-abelian gauge field*. This theory has afterwards gone through a tremendous development and is today best understood using the theory of fiber bundles where it is characterized as a principal bundle. Here the theory is characterized with the quadruple (M, P, G, π), where M is the base manifold, P is the total space, even called the principal bundle itself, G is the gauge group and π is a projection map which is a surjective submersion characterized by the exactness of the following short sequence

$$0 \longrightarrow G \longrightarrow^{i} P \longrightarrow^{\pi} M \longrightarrow 0 \tag{2.1}$$

This sequence can serve as the definition of a fibration and implies that $M = P/G$. This relation can locally be inverted to say that locally $P = M \times G$, but globally this is not always true - one can have obstructions which in the physical language are called magnetic monopoles or instantons depending on what character they have. Next chapter will deal with these concepts further. Mathematically these obstructions is measured by characteristic classes. That is to say that there are characteristic classes which are topologically invariant objects and can therefore see topological differences between two different principal bundles with same base manifold and gauge group. To be exact it should be stressed that these characteristic classes require the introduction of a connection by which they are characterized as polynomials in the curvature tensor associated with this connection. They are integral classes and thus to every principal bundle there is an associated integer, so two different integers implies that the two principal bundles are non-homeomorphic. This does not say though, that two principal bundles with the same integer associated with some characteristic class, must be homeomorphic. A connection in a fiber bundle basically means a splitting of the tangent bundle into two different parts, often referred to as the horizontal and vertical parts, and one writes $TP = H \oplus V$. The connection now measures the twisting of the gauge group as one moves across the base manifold. This can be seen as the non-integrability of the horizontal subspace. We can generalize this further by introducing what is called an almost product structure (APS) which is a $(1,1)$ tensor that splits the manifold into $TM = E_+ \oplus E_-$. This tensor together with a compatible metric on TM gives a total description of the geometry and topology of the manifold in question. Curvature as introduced in fiber bundle theory will find its generalization in the Nijenhuis tensor of the almost product structure in question. Paper IV and V makes a deep dissection into these areas in which the innermost structure of this type, by which fiber bundles is just a special case, is investigated. For the interested reader other beneficial input is found in Refs. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

2.1 Basics regarding almost product manifolds

An almost product manifold is a triple (M, g, I) where M is a manifold, g is a metric compatible with the almost product structure I, i.e., $g(X, Y) = g(IX, IY)$ with $I^2 = \mathbb{1}$. Together with the Levi-Civita connection will the almost product structure give a most thorough description of the geometry and topology of an almost product manifold and therefore also of fiber bundles. There are two characteristic tensors which are called deformation tensors of the (respective) distribution associated with the almost product structure. There are also, beside the Levi-Civita connection, two other covariant derivatives with the property that they commute with the APS. All these covariant derivatives can be seen to differ only through these deformation tensors. These deformation tensors are (2,1) tensors with the property that two of the indices lies in one of the distributions and the other index lies entirely in the complementary one. One can reduce the tensors by splitting the two indices into their anti-symmetric, symmetric and traceless and trace parts respectively. But let us first get to know the concept of almost product structures a little better. The APS makes a split of the tangent bundle in the following way

Definition 2.1 *Let I be an almost product structure on $\underline{\mathscr{M}}$, then I defines two natural distributions of $T\underline{\mathscr{M}}$, denoted $\mathcal{D}$ and $\mathcal{D}'$ respectively, in the following way. Let*

$$
\begin{aligned}
\mathcal{D}_x &:= \{X \in T_x\underline{\mathscr{M}} : IX = X\} \\
\mathcal{D}'_x &:= \{X \in T_x\underline{\mathscr{M}} : IX = -X\}
\end{aligned}
$$

then

$$\mathcal{D} := \bigcup_{x \in \underline{\mathcal{M}}} \mathcal{D}_x, \quad \mathcal{D}' := \bigcup_{x \in \underline{\mathcal{M}}} \mathcal{D}'_x$$

Here the adaption to the conventions of paper IV should be stressed. According to the above definition, the tangent bundle now splits into

$$T\underline{\mathcal{M}} = \mathcal{D} \oplus \mathcal{D}'. \tag{2.2}$$

These complementary distributions can of course be associated as the horizontal and vertical in ordinary fibre bundle theory. Projections of vectors in the tangent bundle onto these respective distributions can be made by introducing projection operators through the APS. This is of course possible because the APS squares to one.

$$\mathcal{P} \;:=\; \frac{1}{2}(\,\mathbb{1} + I) \tag{2.3}$$

$$\mathcal{P}' \;:=\; \frac{1}{2}(\,\mathbb{1} - I) \tag{2.4}$$

$$\tag{2.5}$$

As the manifold studied here is endowed with a metric compatible with the APS, this metric is split into the parts of the two complementary distributions respectively, i.e.

$$\underline{g}(X,Y) = g(X,Y) + g'(X,Y). \tag{2.6}$$

where $g(X,Y) := \underline{g}(\mathcal{P}X, \mathcal{P}Y)$ and $g'(X,Y) := \underline{g}(\mathcal{P}'X, \mathcal{P}'Y)$. Associated with these complementary distributions are their respective deformation tensor which can be said to measure the failure of the split of the tangent bundle to split the entire manifold into a geometrically direct product. By "geometrically" it is stressed that the presence of the metric has been taken into consideration. Should the metric be absent one could still talk about a direct product which would be purely of topological origin. The Nijenhuis tensor, yet to be introduced, measures this failure and will be seen to represent the gauge field strength in a principal bundle. Of course, as is notably the case for a principal bundle, nothing depends on a possible fiber metric so it is of no interest to talk about internal geometry there. The tensor that sees the geometrical differences will be the Jordan tensor and can be defined in a similar but yet other way to the Nijenhuis tensor. To understand this better the deformation tensor first needs to be defined. Only one of the deformation tensors will be regarded but the other is analogous.

Definition 2.2 *Let $\mathcal{D}$ be a k-distribution with projection $\mathcal{P}$ on a riemannian manifold $\mathcal{M}$ with non-degenerate metric g. Let ∇ be the Levi–Civita connection with* *respect to this metric and let $\mathcal{P}' := \mathbb{1} - \mathcal{P}$ be the coprojection of $\mathfrak{D}$. Now define the following tensors with characteristics*

$$\begin{aligned} H, L, K: &\quad \Lambda^1_{\mathfrak{D}} \times \Lambda^1_{\mathfrak{D}} \longmapsto \Lambda^1_{\mathfrak{D}'} \\ \kappa: &\quad \Lambda^1_{\mathfrak{D}'} \longmapsto \mathbb{R} \end{aligned}$$

and

(i)	$H(X,Y) := \mathcal{P}'\nabla_{\mathcal{P}X}\mathcal{P}Y$	**deformation tensor**,
(ii)	$L(X,Y) := \frac{1}{2}(H(X,Y) - H(Y,X))$	**twisting tensor**,
(iii)	$K(X,Y) := \frac{1}{2}(H(X,Y) + H(Y,X))$	**extrinsic curvature tensor**,
(iv)	$^\sharp\kappa := \mathrm{tr}H$	**mean curvature tensor**,
(v)	$W(X,Y) := K(X,Y) - \frac{1}{k}{}^\sharp\kappa g(X,Y)$	**conformation tensor**.

This gives us the decomposition of the deformation tensor in its anti-symmetric, symmetric-traceless and trace parts accordingly,

$$H(X,Y) = L(X,Y) + W(X,Y) + \frac{1}{k}{}^\sharp\kappa g(X,Y).$$

From this definition it is clear that the deformation tensor measures to what extent the distribution deforms into the complementary distribution under parallel transport. Here parallel transport is defined by the usual Levi–Civita connection defined by

$$\nabla\varphi(X,Y) := \frac{1}{2}(d\varphi(X,Y) + \mathscr{L}_{^\sharp\varphi}g(X,Y)) \tag{2.7}$$

In the case where no metric is present there is no unique way to introduce parallel transport but one has to use the dragging of vector fields instead. This is of course nothing but the Lie derivative and by substituting the Levi–Civita connection by the Lie derivative in the definition above, the usual curvature of a fibration is in fact obtained. It also follows by the no-torsion condition of the Levi–Civita connection that the anti-symmetric part of the deformation tensor called the twisting tensor indeed measures this curvature.

$$L(X,Y) = \frac{1}{2}\mathcal{P}'[\mathcal{P}X, \mathcal{P}Y] \tag{2.8}$$

Recall the integrability condition for a distribution to be a foliation which states that the commutator of vectors in the distribution must stay in the distribution, or $[\Lambda_{\mathfrak{D}}, \Lambda_{\mathfrak{D}}] = \Lambda_{\mathfrak{D}}$. This turns out to be nothing but the requirement that this twisting tensor vanishes. The remaining part of the deformation tensor has conventionally been called the extrinsic curvature as it in next section will be identified with the extrinsic curvature of an embedding regarded as a leaf of a foliation. As the twisting tensor does not depend on the metric at all, it is clear that the remaining part must do. In fact, if one looks at the very definition of the Levi–Civita connection it is of no surprise that the extrinsic curvature can be written

$$K(X,Y)(\varphi) = -\frac{1}{2}\mathscr{L}_{\sharp\varphi'}g(X,Y), \;\; or \quad {}^{\flat}K(X,Y,Z) = -\frac{1}{2}\mathscr{L}_{Z'}g(X,Y), \tag{2.9}$$

where the prime denotes projection along the normal directions by $\mathcal{P}'$. This relation gives a most geometrical insight in what the extrinsic curvature measures. As is seen as one moves in some complementary direction the extrinsic curvature measures the failure of this to be an isometry of the induced metric on the distribution. So it is clear that the extrinsic curvature sees how the manifold is geometrically deformed as one moves in a complementary direction while the twisting tensor sees how the complementary distribution twists as one goes around a loop in the distribution. In the definition of the deformation tensor the extrinsic curvature was split further into its irreducible parts. The reason for this is that only the mean curvature which represents the trace part of the deformation tensor sees a conformal transformation. If one makes a conformal transformation with ${}^{c}\underline{g} = \lambda\underline{g} = e^{2\phi}\underline{g}$ the tensors transforms as

$$\begin{aligned}
{}^{c}K(\varphi) &= K(\varphi) + \lambda^{-1}\sharp\varphi'[\lambda]g = K(\varphi) + 2\sharp\varphi[\phi]g && (2.10)\\
{}^{c}\kappa(X) &= \kappa(X) + k\lambda^{-1}X'[\lambda] = \kappa(X) + 2kX'[\phi] && (2.11)\\
{}^{c}W &= W && (2.12)\\
{}^{c}L &= L && (2.13)
\end{aligned}$$

From these relations it is clear that the conformation tensors measures geometrical deformations which preserve the volume, while the mean curvature represents the blow up of the manifold while moving in a complementary direction. From the theory of embeddings it is of no surprise that a distribution with vanishing mean curvature tensor will be called minimal as this is the condition which minimizes the volume functional of the induced metric. A simple but yet very instructive calculation shows the origin of this condition from the action functional, namely

$$\delta\int d^k x\sqrt{g} = \int d^k x\sqrt{g}g^{mn}\delta g_{mn} \tag{2.14}$$

Now generating a special variation by a normal vector $\delta_{X'}g_{mn} = (\mathscr{L}_{X'}g)_{mn} = {}^{\flat}K_{mn}(X')$ which by the vanishing of the variation gives $\kappa(X') = 0, \forall X' \in \Lambda^1_{\mathfrak{D}'}$ and it is clear that a minimal distribution requires the vanishing of the mean curvature tensor. This volume functional is nothing but the action describing the dynamics of a p-brane in various supergravity theories, which will be discussed in chapter 6. So by the irreducible parts of the deformation tensor one extracts eight different situations.

Figure 2.1: Overview of the different classes of a distribution

Definition 2.3 *Let $\mathcal{D}$ be a distribution on a riemannian manifold $\mathscr{M}$ we have the following 8 different classes*

Name	$L = 0$	$W = 0$	$\kappa = 0$	*Notation*
Distribution				D
Minimal Distribution			x	MD
Umbilic Distribution		x		UD
Geodesic Distribution		x	x	GD
Foliation	x			F
Minimal Foliation	x		x	MF
Umbilic Foliation	x	x		UF
Geodesic Foliation	x	x	x	GF

If one takes into account the second deformation tensor associated with the complementary distribution and observe that as the APS is symmetric in the sense that changing I to $-I$ still gives the same splitting although now with positive and negative eigenspaces interchanged, one gets a total of $36 = \frac{1}{2}8(8+1)$ different classes.

Proposition 2.4 *Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ be an riemannian almost product struc-*

ture. We then have the following 36 different classes

Classes	L	W	κ	L'	W'	κ'	*Name*
(GF,GF)	x	x	x	x	x	x	*Local product*
(GF,UF)	x	x	x	x	x		*Twisted product*
(GF,MF)	x	x	x	x		x	
(GF,F)	x	x	x	x			
(UF,UF)	x	x		x	x		*Double twisted product*
(UF,MF)	x	x		x		x	
(UF,F)	x	x		x			
(MF,MF)	x		x	x		x	
(MF,F)	x		x	x			
(F,F)	x			x			
(GF,GD)	x	x	x		x	x	*Riemannian foliation*
(UF,GD)	x	x			x	x	*Riemannian foliation*
(MF,GD)	x		x		x	x	*Riemannian foliation*
(F,GD)	x				x	x	*Riemannian foliation*
(GF,UD)	x	x	x		x		
(UF,UD)	x	x			x		
(MF,UD)	x		x		x		
(F,UD)	x				x		
(GF,MD)	x	x	x			x	
(UF,MD)	x	x				x	
(MF,MD)	x		x			x	
(F,MD)	x					x	
(GF,D)	x	x	x				
(UF,D)	x	x					
(MF,D)	x		x				
(F,D)	x						
(GD,GD)		x	x		x	x	
(GD,UD)		x	x		x		
(GD,MD)		x	x			x	
(GD,D)		x	x				
(UD,UD)		x			x		
(UD,MD)		x				x	
(UD,D)		x					
(MD,MD)			x			x	
(MD,D)			x				
(D,D)							

Here the ordinary case of a principal bundle is simply (GF, GD) where the non-integrable distributions is the tangentbundle of the base manifold. This case can be seen as having one base manifold and one fiber manifold and starting to twist the fiber as we move along directions on the base manifold. This twisting imposes a non-integrability to the base manifold itself. In fiber bundle theory one asks how many topologically different total spaces can be made starting from one base manifold and one fiber. From the above analysis it is clear that, although in the classification we did not distinguish the class (GF, GD) from (GD, GF), there is an asymmetry whether one twists the fiber above the base manifold or "twists the base manifold over the fiber". This gives yet another possibility of topological obstructions due to the twisting of the Minkowski part of the total space (previously the base manifold) over the complementary space (previously the fiber). These obstructions would have no counterpart in existing physics where we thought of them as monopoles or instantons. Nevertheless they must exist as a matter of symmetry of the classification and the ignorance of the Einstein equations to distinguish between these cases so from above analysis one might conjecture their existence. The physical interpretation will be left to the person who gives us the first solution of that kind. It should also be stressed that although given an almost product manifold $(M, \underline{g}, I)$ with given metric and given APS it will of course fit into one of these classes but when studying manifolds one might ask whether the manifold itself admits a metric in some of these classes given an APS. One might also ask what kinds of APS there is on a manifold. As the APS squares to one it is obvious that it only have eigenvalues ± 1 and $trI = 2k - m$ where k is the dimension of the positive eigenspace (distribution) and $k' = m - k$ is the dimension of the negative eigenspace (complementary distribution). The first question to ask is therefore whether the manifold admits an APS with given k. The easiest example of possible obstructions is the case $k = 1$ which in minkowskian physics would refer to a point particle. This question can easily be answered because the condition for globally defined distribution of dimension one, which always is integrable, is the same as the existence of a global vectorfield. Now the existence of a global vectorfield is equivalent to the condition that the euler number of the manifold is zero. This is also the condition for the existence of a codimension one foliation and the condition for the existence of a metric with minkowskian signature. This tells us for example that there always exist an APS with $k = 1$ on a minkowskian manifold and the existence of an integrable complementary distribution gives us the possibility to do the classical ADM decomposition when doing canonical gravity.

In the previous analysis the APS gave us two distinct eigenspaces to which we associated two different deformation tensors, but the analysis did not make explicit use of the APS itself. So here will be given a briefing about how one can approach all distinct tensors from the APS itself. For a complete analysis of how one deals with 1-1 tensors algebraically see paper IV. As is well known, the exterior algebra of contravariant fields deals with commutators of vectorfields while the exterior algebra of forms deals with the exterior derivative. Now to a given 1-1 tensor there is associated what is called an I-bracket where I stands for the 1-1 tensor at hand, in our case of course an APS. This is defined by

$$[X, Y]_I := [IX, Y] + [X, IY] - I[X, Y] \tag{2.15}$$

and can be seen to be anti-symmetric and to reduce to the ordinary bracket for the identity, $1\!\!1$. One can define the Nijenhuis tensor as the failure of this I-bracket to be a Lie bracket, but there is yet another, more common approach to the definition of the Nijenhuis tensor, namely from the exterior derivative of forms. Here one instead defines $d_I := [d, i_I]$ as a generalized Lie derivative and then defines the Nijenhuis tensor as the failure of this new operator to be a co-boundary operator. That is the failure of this operator to be nilpotent, so the more common Nijenhuis tensor reads $< -N_I(X,Y), df >:= d_I d_I f(X,Y)$. The sign which actually is non-standard is convenient for the comparison with algebraic gauge theory and as it turns out when taken into account the two new covariant derivatives associated with an APS the Nijenhuis tensor can be identified with the torsion or parts of it for those non-Levi-Civita covariant derivatives. By the definition of the I-bracket the Nijenhuis tensor can be written in its most pleasant form.

$$N_I[X,Y] := I([X,Y]_I) - [I(X), I(Y)], \tag{2.16}$$

From this relation one can regard the Nijenhuis tensor as a measure of to what extent the endomorphism, I, fails to be a Lie algebra homomorphism of the infinite dimensional Lie algebra of vectorfields on the manifold. In algebraic gauge theory we have a principal bundle $0 \to A \to E \to B \to 0$ and a connection $\rho : B \to E$ with curvature

$$F(X,Y) := \rho([X,Y]|_B) - [\rho(X), \rho(Y)]|_E. \tag{2.17}$$

Here the curvature measures the failure of the connection ρ to be a Lie algebra homomorphism as a map from the base algebra B to the total algebra E. In the case with the endomorphism I the map is from the total space into the total space. By a simple example one can show that one can lift the concept of curvature of the algebraic gauge theory entirely into the total space itself. The connection on B denoted ρ is map such that

$$\rho : B \longmapsto E, \qquad \pi \circ \rho = 1\!\!1_B. \tag{2.18}$$

But we could equivalently look at a connection in the total space, E, instead and denote it by ω where it instead satisfies

$$\omega : E \longmapsto A, \qquad \omega \circ i = 1\!\!1_A. \tag{2.19}$$

There is an immediate relation between the two connections given by

$$\omega = 1_E - \rho \circ \pi \tag{2.20}$$

satisfying the idempotency condition $\omega^2 = \omega$ of a projection operator which is the key connection to the APS structure. The curvature of these two connections are defined for $X, Y \in \Lambda^1_B$ and $\underline{X}, \underline{Y} \in \Lambda^1_E$ by

$$\begin{aligned} F(X,Y) &:= \rho([X,Y]) - [\rho(X), \rho(Y)], && (2.21)\\ \Omega(\underline{X}, \underline{Y}) &:= F(\pi\underline{X}, \pi\underline{Y}). && (2.22) \end{aligned}$$

Introducing a co-projection operator $\omega' = \mathbb{1}_E - \omega = \rho \circ \pi$ we can now fully extend this theory to the APS case by introducing $I = \omega - \omega' = 1_E - 2\rho \circ \pi$ and with the curvature Ω reducing to the Nijenhuis tensor of this APS,

$$\Omega(\underline{X}, \underline{Y}) = \frac{1}{4} N_I(\underline{X}, \underline{Y}) \tag{2.23}$$

which follows directly when expressing Ω in terms of ω, see [12]. So in conclusion we notice the naturality of the Nijenhuis tensor as a measure of twisting in fiber bundles and likewise in the generalization to almost product manifolds. In principal bundle theory we know from the Ambrose-Singer theorem that the curvature lies in the Lie algebra of the holonomy group of the principal bundle. This holonomy group is by elementary group theory forced to be a subgroup of the gauge group of the theory, generically the entire gauge group itself. In the case of an almost product manifold it is not that easy because the existence of an APS on a manifold does not imply that it can be seen as two locally product manifolds twisting around each other. Although in the (GF, GD) case, when we have this situation and locally the manifold looks like $\underline{M} = \mathscr{M} \times \mathscr{M}'$, the Nijenhuis tensor lies in the Lie algebra of the holonomy group which now is a subgroup of the diffeomorphism group $Diff(\mathscr{M}')$ with Lie algebra the set of vectorfields on $\mathscr{M}'$. In the case of a doubly non-integrable almost product manifold for instance (GD, GD), where the manifold again can be interpreted as a double twisting of $\mathscr{M}$ and $\mathscr{M}'$ around each other, the Nijenhuis tensor will have two non-vanishing parts which are nothing but the two twisting tensors of the respective distribution, i.e.

$$\frac{1}{8} N_I = -L - L' \tag{2.24}$$

This relation tells us that the Nijenhuis tensor indeed measures the twisting of the two complementary distributions associated with the APS. It is also clear from the definition that the Nijenhuis tensor is independent of a particular metric on the manifold in question, so to wrap up the anti-symmetric part of the deformation tensor is metric independent and can be seen to be measured by the Nijenhuis tensor.

In physical theories including gravity we are endowed with a metric, and in particular in Kaluza–Klein theories where gauge theory is looked upon from a total space point of view. It is therefore of utmost importance to study not only twistings which are done by the Nijenhuis tensor but also geometrical differences which are connected to the symmetric part of the deformation tensor as seen before. This study can too be done through the APS directly. So if we start by defining the Jordan bracket by

$$\{X, Y\} := \underline{\nabla}_X Y + \underline{\nabla}_Y X \tag{2.25}$$

and in a similar fashion as with the I-bracket define the I-Jordan bracket

$$\{X, Y\}_I := \{IX, Y\} + \{X, IY\} - I\{X, Y\}. \tag{2.26}$$

we can define a new tensor, called the Jordan tensor, in exactly the same way as we did with the Nijenhuis tensor earlier, namely

$$M_I(X,Y) := I\{X,Y\}_I - \{IX,IY\} \tag{2.27}$$

Again it should be pointed out that when choosing $I = 1\!\!1$ the I-Jordan bracket reduces to the ordinary Jordan bracket as was the case for the I-bracket and at the same time the Jordan tensor vanishes which of course also is the case for the Nijenhuis tensor. So as there is such a great similarity to the Nijenhuis tensor case it would come with no surprise when one calculates the Jordan tensor in terms of the extrinsic curvatures and finds out that one has

$$\frac{1}{8}M_I = -K - K' \tag{2.28}$$

So one realizes that as the non-integrable twisting behavior which was non-metric dependent was measured by the Nijenhuis tensor the geometrical deformations which solidly depends on the metric is measured by the Jordan tensor in the APS language. One can of course put this together into a total deformation tensor given entirely in the APS language as

$$H_I(X,Y) := N_I(X,Y) + M_I(X,Y) \tag{2.29}$$

Interesting to note here is that this can serve as a definition of the deformation tensor of an arbitrary endomorphism I and not only APS. For instance every studied object in almost product manifolds has its counterpart in the complex case where one have an almost complex structure which squares to minus one instead of one. All these tensors look completely the same when expressed in terms of the endomorphism I but the interpretation in terms of some extrinsic curvature tensors and twisting tensors might have to be reinterpreted.

As previously mentioned, there are two other covariant derivatives beside the Levi-Civita connection which are of severe interest in almost product manifolds and consequently in Kaluza-Klein theory and gauge theory. These are the only two other connections with a natural geometrical origin that commute with the almost product structure. Before making the precise definitions in terms of the APS it is most instructive to start by taking a look at the Levi-Civita connection expressed in the oriented frame. The oriented frame is defined by imposing ortho-normality to the basis vectors which bring them down to vielbeins and a $SO(m)$ degree of arbitrariness and then forcing them to be eigenvectors of the APS which breaks the $SO(m)$ arbitrariness down to $SO(k) \times SO(k')$. So if we denote $E_{\bar{a}} = (E_a, E_{a'})$ the ortho-normal eigenvectors of I, *i.e.*, $IE_a = E_a, IE_{a'} = -E_{a'}$, and denote the connection 1-form with respect to this basis conventionally by $\underline{\omega}$, we have from the definition of the Levi-Civita connection $\nabla_{\bar{a}} E_{\bar{b}} =: \underline{\omega}_{\bar{a}\bar{b}}{}^{\bar{c}} E_{\bar{c}}$. Recall the definition of the deformation tensor which in this oriented basis take a most pleasant form $H_{ab}{}^{c'} := \mathcal{P}'\underline{\nabla}_a E_b = \underline{\omega}_{ab}{}^{c'} E_{c'}$ where for the first time the true index structure of deformation tensor is explicitly revealed. If one in addition denotes the normal connections by Ω, *i.e.*, $\underline{\omega}_{ab'}{}^{c'} =: \Omega_{ab'}{}^{c'}$, the Levi-Civita connection can be written

$$\underline{\omega} = \left[\begin{pmatrix} \omega & H \\ -H^t & \Omega \end{pmatrix}, \begin{pmatrix} \Omega' & H' \\ -H'^t & \omega' \end{pmatrix} \right] \tag{2.30}$$

where the index structure $\underline{\omega}_{\bar{a}\bar{b}}{}^{\bar{c}} = [{}_a()_{\bar{b}}{}^{\bar{c}}, {}_{a'}()_{\bar{b}}{}^{\bar{c}}]$ is understood. From this expression it is more transparent what was already clear from the very definition of the deformation tensors, that the deformation tensors are off-diagonal parts of the Levi-Civita connection. Now it is also clear that the Levi-Civita connection does not commute with the APS unless both these deformation tensors are zero. It is also clear from connection analysis that if one introduces yet another connection just by subtracting the deformation tensor parts one ends up with a new connection which, although not Levi-Civita, must commute with the APS because it evidently preserves the rigging. This is also the first of the two new connection which were promised earlier and it is dubbed the adapted connection due to its adaption to the rigging associated with an APS. In the next section the analogy with the induced and normal connections of an embedding will be seen, where one compares them with the adapted connection through the Gauss-Weingarten equations. Before defining the adapted connection axiomatically through the almost product structure it must be stressed that adding a tensor to a connection does of course alter some of its structure. When compared to the Levi-Civita connection one can pay the price of torsion or the price of non-metricity. As is actually clear from the above structure of the Levi-Civita connection one can not alter the metric compatibility of the connection by removing the off diagonal terms so we can, without doing the explicit calculation, tell that the adapted connection is metric compatible but must have a non-vanishing torsion. This torsion will evidently also be zero in the case when the deformation tensors both are zero. Put in the words of APS the following definition arises.

Definition 2.5 *Let $\underline{\mathscr{M}}$ be a riemannian or pseudo-riemannian manifold with non-degenerate metric $\underline{g}$ and corresponding Levi–Civita connection $\underline{\nabla}$. Let I be an almost product structure, then the following two definitions of the adapted connection are equivalent*

(i). $\tilde{\underline{\nabla}}_X Y := \underline{\nabla}_X Y + A(X,Y), \quad A(X,Y) := \frac{1}{2} I \underline{\nabla}_X I(Y)$

(ii). $\tilde{\underline{\nabla}}_X Y := \mathcal{P}\underline{\nabla}_X \mathcal{P}Y + \mathcal{P}'\underline{\nabla}_X \mathcal{P}'Y$

Working out the component expression of the tensor A one ends up exactly in the form anticipated in the above analysis.

$$A_{\bar{a}\bar{b}}{}^{\bar{c}} = \left[\begin{pmatrix} 0 & -H_{ab}{}^{c'} \\ -H_{ab'}{}^{c} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -H'_{a'b'}{}^{c} \\ -H'_{a'b}{}^{c'} & 0 \end{pmatrix} \right] \tag{2.31}$$

This leaves us with a connection form for the adapted connection denoted by $\underline{\tilde{\omega}}$ in the oriented basis looking like

$$\underline{\tilde{\omega}} = \left[\begin{pmatrix} \omega & 0 \\ 0 & \Omega \end{pmatrix}, \begin{pmatrix} \Omega' & 0 \\ 0 & \omega' \end{pmatrix} \right] \tag{2.32}$$

Due to the absence of the off-diagonal parts this connection was seen to commute with the almost product structure, and as was noted before it was also metric compatible. Put together this implies that the adapted connection is metric with respect to the induced metrics on the the two subbundles defined by the almost product structure, i.e. put together the adapted connection satisfies

$$\begin{aligned} \underline{\tilde{\nabla}}\underline{g} &= 0 \\ \underline{\tilde{\nabla}} I &= 0 \\ \underline{\tilde{\nabla}} g &= 0 \\ \underline{\tilde{\nabla}} g' &= 0 \end{aligned}$$

It is the property that adapted connection is metric compatible not only with the total metric but also with the induced metrics that makes it so useful while studying a rigging like the one defined by the APS. The price we had to pay was that this connection was not torsion-free but depended on the deformation tensors. Previously it was mentioned that there was some sort of connection between the torsion and the Nijenhuis tensor. The connection in terms of the torsion of the adapted connection is

$$\frac{1}{2} N_I(X, Y) = \underline{\tilde{T}}(X, Y) + \underline{\tilde{T}}(IX, IY) \tag{2.33}$$

Although there obviously is a connection it is not in that pleasant form one most would like. As is obvious from the definition of the torsion tensor in terms of Cartan's structure equation $dE^a + \omega_b{}^a E^b = T^a$ and Frobenius theory of integrability, the torsion could in some sense be regarded as a measure of the non-integrability of the vielbeins. If one considers the complete set of vielbeins of the total manifold these are by definition integrable and the torsion tensor of a connection is in that case only a measure of a "badly chosen connection". (There are of course situations where it even in this case is preferable to choose a connection with torsion instead of the Levi-Civita connection). In the case with an APS present it is rather different, here the split of the vielbeins into the oriented base associated with the APS forces one to split the Cartan's structure equations into two different parts. These parts can then be seen as a measure of the non-integrability of the respective distributions spanned by their associated vielbeins. Just by looking at Cartan's structure equation it is obvious that just by moving the off-diagonal parts of the Levi-Civita connection to the right and identifying them as the torsion of the new diagonal connection. This is actually precisely the adapted connection that is received this way, but by of both Frobenius integrability condition and the Cartan's structure equation it would be pleasant to find another connection where the torsion exactly measures the non-integrability of respective distributions locally spanned by the oriented vielbeins. This is indeed possible and the connection known as the Vidal connection is defined in terms of the APS as follows.

Definition 2.6 *Let $\underline{\mathscr{M}}$ be a riemannian or pseudo-riemannian manifold with non-degenerate metric $\underline{g}$ and corresponding Levi–Civita connection $\underline{\nabla}$, let I define a foliation in the previous sense, then the Vidal connection is defined by*

$$\tilde{\tilde{\underline{\nabla}}}_X Y := \tilde{\underline{\nabla}}_X Y + B(X,Y), \quad B(X,Y) := \frac{1}{4}(\underline{\nabla}_{IY} I + I\underline{\nabla}_Y I)(X)$$

It is of course obvious that as the tensors A was given in terms of the deformation tensors, so must be the case with this new tensor B. In fact they can be proved to be related.

$$B(X,Y) = \frac{1}{2}\left(A(Y,X) - A(IY,IX)\right). \tag{2.34}$$

The shifting of the vectorfields in the respective tensors in the above equation stresses the change in the matrix structure of the tensor B compared to A. In the earlier introduced matrix form the tensor B reads

$$B_{\bar{a}\bar{b}}{}^{\bar{c}} = \left[\begin{pmatrix} 0 & 0 \\ 0 & -H'_{b'a}{}^{c'} \end{pmatrix}, \begin{pmatrix} -H_{ba'}{}^{c} & 0 \\ 0 & 0 \end{pmatrix}\right] \tag{2.35}$$

The new structure of the Vidal connection that makes its torsion depend only on the Nijenhuis tensor is based solidly on the fact that the structure constants with index form $C_{ab'}{}^{c'}$ and $C_{a'b}{}^{c}$ transform as connections under local $SO(k) \times SO(k')$ transformations. By the torsion equation of the Levi-Civita connection taken for the above index structures on gets $0 = \omega_{ab'}{}^{c'} - \omega_{b'a}{}^{c'} - C_{ab'}{}^{c'} \Rightarrow \Omega_{ab'}{}^{c'} - H'_{b'a}{}^{c'} = C_{ab'}{}^{c'}$. So in matrix form in the oriented frame the Vidal connection form reads

$$\tilde{\tilde{\underline{\omega}}} = \left[\begin{pmatrix} \omega & 0 \\ 0 & C \end{pmatrix}, \begin{pmatrix} C' & 0 \\ 0 & \omega' \end{pmatrix}\right] \tag{2.36}$$

The diagonal structure of this connection ensures the feature that it commutes with the almost product structure, i.e.,

$$\tilde{\tilde{\underline{\nabla}}}_X I = 0 \tag{2.37}$$

As was discussed earlier this construction made it possible for the torsion of the Vidal connection only to measure the non-integrability of the respective distributions. It is therefore clear that the torsion can be written

$$\frac{1}{4} N_I(X,Y) = \tilde{\tilde{\underline{T}}}(X,Y) \tag{2.38}$$

Now it is time to make some honest confessions regarding the Vidal connection. As was clear when studying the adapted connection the price we had to pay for it to commute with the APS was that it picked up torsion components. Here in the Vidal connection we have completely hidden the structure of the deformation tensors appearing in the Levi-Civita connection. The price to be paid was of course a torsion tensor but this torsion only have components equal to the Nijenhuis tensor which measure the non-integrability but no components related to the other parts of the deformation tensor. So to restore the information contained in these parts there is merely only one other place to look namely in the metricity condition. The Vidal connection will turn out to be non-metric compatible in the case when any of the extrinsic curvature parts is different from zero. In fact the following relation holds

$$\frac{1}{8}\underline{g}(X, M_I(Y,Z)) = (\underline{\tilde{\tilde{\nabla}}}_X \underline{g})(Y,Z) \tag{2.39}$$

So put together the Vidal connection have one non-metricity part directly connected to Jordan tensor and one torsion part directly connected to the Nijenhuis tensor.

2.2 Comparison with embeddings

When comparing embeddings and almost product structures in a manifold, the very first distinction between these is that an embedding have lower dimension and will just be a very tiny subset of the points of the target space. The source manifold will throughout this thesis be denoted world-sheet no matter what the dimension it has and all manifolds treated are supposed to be equipped with a metric. Take into account an arbitrary embedding of a manifold $(\mathscr{M}, h)$ with dimension k into another manifold $(\underline{\mathscr{M}}, \underline{g})$ with dimension $\underline{m}$ and introduce local coordinates x^m and $x^{\underline{m}}$ respectively[1]. The tangent spaces $T\mathscr{M}$ $(T\underline{\mathscr{M}})$ will be spanned by the coordinate basis ∂_m $(\partial_{\underline{m}})$. The tangent space of $T\mathscr{M}$ embedded in $T\underline{\mathscr{M}}$ will be spanned by the vectors $f_*\partial_m$. Here a basic distinction between almost product structures is that this hypersurface associated with an embedding must indeed be integrable while the APS restricted to the same hypersurface need not. In a non-coordinate basis chosen to be an orthonormal basis, so called vielbeins $\underline{E_{\underline{a}}}$ can be introduced in target space. These are determined up to a local $O(\underline{m})$ transformation. As the $f_*\partial_m$ will span a k dimensional vector subspace of $T_{\underline{x}}\underline{\mathscr{M}}$, it is always possible to introduce a new frame of orthonormal basis vectors in such a way that $\underline{E}_a$ will point in the tangent directions of the embedding and $\underline{E}_{a'}$ will point in the normal directions of the embedded surface. The normal directions are those metrically orthogonal to the $\underline{E}'_a s$. These basis vectors are decided up to local $O(m)$ and $O(\underline{m}-m)$ rotations respectively. The quotient of this is the well known grassmanian

$$Gr(k, \underline{m}) = \frac{O(\underline{m})}{O(k)\times O(\underline{m}-k)} \tag{2.40}$$

[1] So far the signature of the manifolds are of no importance, so one can see the dimensions as $m = (p, m-p)$ and $\underline{m} = (\underline{p}, \underline{m}-\underline{p})$.

of k-planes in $\underline{m}$ dimensions. Just for convenience we will denote the normal dimension by $k' := \underline{m} - k$. This can be compared to the structure of an APS. Here one can find orthonormal basis vectors which are the so called oriented frame which are positive and negative eigenvalues of the APS respectively, i.e. $I\underline{E}_a = \underline{E}_a, I\underline{E}_{a'} = -\underline{E}_{a'}$. One can equally say that the APS, I, breaks the structure group $O(\underline{m})$ of $T\underline{\mathscr{M}}$ down to $O(k) \times O(k')$ where $k' := \underline{m} - k$. In that sense the set of almost product structures with k positive eigenvalues is parameterized by the grassmannian,

$$I \in Gr(k, \underline{m}) = \frac{O(\underline{m})}{O(k) \times O(k')} \tag{2.41}$$

with the basic difference that this structure is now globally defined throughout the entire manifold and not restricted to the subset of points associated to a embedded manifold. The grassmannian has $kk' = k(\underline{m} - k)$ independent components and parameterizes the space of k-planes in $\mathbb{R}^{\underline{m}}$. The almost product structure thus defines two complementary distributions by taking these complementary hyperplanes spanned by the eigenfunctions with positive eigenvalues and by the eigenfunctions with negative eigenvalues respectively. Put together these basis vectors form a total basis for $T\underline{\mathscr{M}}$ we get

$$\underline{E}_{\bar{a}} := (\underline{E}_a, \underline{E}_{a'}) \tag{2.42}$$

and the extension of the local transformations by $O(m)$ and $O(n)$ will look like

$$O(m) \rightarrow \begin{pmatrix} O(m) & 0 \\ 0 & 1\!\!1_n \end{pmatrix} \qquad O(n) \rightarrow \begin{pmatrix} 1\!\!1_m & 0 \\ 0 & O(n) \end{pmatrix} \tag{2.43}$$

In the case of an embedding though this splitting is only associated to a subset of the tangent space containing the points of the embedded manifold, which can be written $T\underline{\mathscr{M}}\,|_{\mathscr{M}}$. The dual vielbeins can be introduced by the relation

$$\underline{E}^{\bar{a}}(\underline{E}_{\bar{b}}) = \delta_{\bar{b}}{}^{\bar{a}} \tag{2.44}$$

and the canonical 1-form can be seen to be split into its tangential and normal parts

$$\underline{\theta} = \mathcal{P} + \mathcal{P}' \equiv \underline{E}^a \underline{E}_a + \underline{E}^{a'} \underline{E}_{a'}. \tag{2.45}$$

In this basis the almost product structure looks like

$$I = \mathcal{P} - \mathcal{P}' \tag{2.46}$$

The induced canonical 1-forms of the respective complementary distribution now serves as projector operators, i.e.

$$\begin{array}{rcl} X^{\|} & = & \mathcal{P}(\underline{X}) \\ X^{\perp} & = & \mathcal{P}'(\underline{X}) \end{array} \tag{2.47}$$

From (2.45) it is obvious that the canonical splitting is invariant under the group $O(m) \times O(n)$ which states that the embedding or the APS is independent of the choice of frame in the two complementary distributions. The embedding/APS is only changed through the grassmanian part of $O(\underline{m})$, i.e. $Gr(k, \underline{m})$. This can be seen by expressing the oriented vielbeins in another arbitrary vielbein base, i.e.

$$\begin{array}{rcl} E_a & = & E_a{}^{\underline{b}} \underline{E}_{\underline{b}} \\ E_{a'} & = & E_{a'}{}^{\underline{b}} \underline{E}_{\underline{b}} \end{array} \tag{2.48}$$

In the composite index form this can be written as

$$\underline{E}_{\bar{a}} = \underline{E}_{\bar{a}}{}^{\underline{b}} \underline{E}_{\underline{b}} = u_{\bar{a}}{}^{\underline{b}} \underline{E}_{\underline{b}} \tag{2.49}$$

where $u_{\bar{a}}{}^{\underline{b}} \in O(\underline{m})$. A general embedding/APS which are parameterized by the grassmanian can be seen through this matrix as

$$u_{\bar{a}}{}^{\underline{b}} = \begin{pmatrix} \left(1\!\!1_k - m^2 \right)^{1/2}{}_a{}^c u_c{}^b & m_a{}^{b'} \\ -u_{a'}{}^{b'} m_{b'}{}^c u_c{}^b & u_{a'}{}^{c'} \left(1\!\!1_{k'} - m^2 \right)^{1/2}{}_{c'}{}^{b'} \end{pmatrix} \tag{2.50}$$

where $m_a{}^{b'}$ spans the grassmanian $Gr(k, \underline{m})$. From this relation it is obvious that the orthogonal group does not split into direct products but that the grassmanian part lies nestled in the orthogonal group. The Lie algebra of the orthogonal group will though split up into direct sums of algebras. The metric on $\underline{\mathscr{M}}$ also splits into two parts

$$\underline{g} = g + g' = \underline{E}^a \underline{E}^b \eta_{ab} + \underline{E}^{a'} \underline{E}^{b'} \eta_{a'b'} \tag{2.51}$$

where g is the intrinsic metric of the embedding or the induced metric of one of the complementary distributions associated with the APS. When studying the intrinsic geometry of an embedding there are two ways to proceed. The first is to do the analogue of the APS and treat it as a subbundle to the tangent bundle of target space. The other possibility is to pull back the metric in target space down to the world-sheet and look at the geometry there. This pullback is of course not possible to do in the case of a pure APS as there is no embedded manifold to pull it back to. The major drawback in this procedure is that all information about the normal bundle is lost during the process. The pullback equations simply looks like

$$\begin{array}{rcl} f^* \underline{g} & = & f^* g =: g \\ f^* \underline{E}^a & =: & E^a \\ f^* g' & = & 0 \\ f^* \underline{E}^{a'} & = & 0 \end{array} \tag{2.52}$$

In the study of embeddings in physics this procedure is nevertheless followed and it is often useful to introduce an auxiliary metric on the world-sheet too. Although the differences between almost product structures and embeddings so far has merely been the restricted location of the embedded surface it must be stressed that almost product structures is not always a generalization of embeddings as it was for gauge theories for example. An embedding is always integrable and its generalization would perhaps be a foliation of the manifold with leaves of the topology of the embedded manifold. This is not always possible as is seen with the simplest case of an embedding of a circle into the plane. If one tries to extend that embedding to a foliation, a singular point arises somewhere when the size of the circle shrinks to zero. If this singular point is removed, though, the extension exists. So in the general case there exist always an extension if one removes some singular points or hypersurfaces.

The differential geometry of embeddings is further studied through the Gauss–Weingarten relations which is a similar split of the Levi–Civita connection as was the case for an APS. Letting tangent vectors be unprimed and normal vectors primed the Gauss–Weingarten relations can be written

$$\begin{aligned} \underline{\nabla}_X Y &= \nabla_X Y + K(X,Y) \\ \underline{\nabla}_X Y' &= \nabla'_X Y' - {}^t K(X,Y') \end{aligned}$$

Here ∇ is a connection on $\mathscr{M}$ compatible with the induced metric g and K is the second fundamental form also known as the extrinsic curvature. For more details see Kobayashi and Nomizu [13]. For a generalization to a non-symmetric connection incorporating torsion see paper II. The connection in $T\underline{\mathscr{M}}$ lies in the Lie Algebra $\mathfrak{o}(\underline{m})$, but the Gauss-Weingarten equations state the following splitting

$$\mathfrak{o}(\underline{m}) = \mathfrak{o}(k) \oplus \mathfrak{o}(k') \oplus \mathfrak{gr}(k,\underline{m}) \tag{2.53}$$

where $\mathfrak{gr}(k,\underline{m})$ is the Lie Algebra of the Grassmannian $Gr(k,\underline{m})$. This splitting is most transparently seen in the basis of the oriented vielbeins.

$$\underline{\nabla}_c \underline{E}_{\bar{a}} = \Omega_{c\bar{a}}{}^{\bar{b}} \underline{E}_{\bar{b}} = \begin{pmatrix} \omega_{ca}{}^{b} & K_{ca}{}^{b'} \\ K_{ca'}{}^{b} & \Omega_{ca'}{}^{b'} \end{pmatrix} \begin{pmatrix} E_b \\ E_{b'} \end{pmatrix} \tag{2.54}$$

This split implies that $\omega_{ca}{}^{b} \in \mathfrak{o}(k), \Omega_{ca'}{}^{b'} \in \mathfrak{o}(k'), K_{ca}{}^{b'} \in \mathfrak{gr}(k,\underline{m})$. In comparison with an APS the embedding must be integrable so the second fundamental form is symmetric. Otherwise it has the same structure as the deformation tensor of an APS. As the normal directions moves out of the embedded surface there is no such associated connection to an embedding. So the second deformation tensor associated with the normal directions is not present here. As the normal directions of course exists in target space one could nevertheless treat those there by doing an APS like splitting restricted to the embedded surface. The geometry in terms of the curvature for an APS which is a natural extension to the embedding case is most thoroughly studied in paper V. Here the comparison is made by identifying the diagonal connection of an embedding, sloppily denoted $\tilde{\nabla} := \nabla + \nabla'$, with the adapted connection of an APS. The diagonal connection is of course the part of the Levi–Civita connection lying in the Lie algebra $\mathfrak{o}(m) \oplus \mathfrak{o}(n)$.

2.3 Comparison with Yang–Mills and Kaluza–Klein theory

The differences between Yang–Mills theory and Kaluza–Klein theory can most easily be seen through the APS theory. Because of the pure geometrical nature of APS theory and the true geometrical origin of Yang–Mills and Kaluza–Klein theory both these theories will be nothing but special cases of the more general APS theory. In light of this the true geometrical differences between Yang–Mills theory and Kaluza–Klein theory can be exploited. Here will be purely focused on the bosonic theory in which Kaluza–Klein theory originates from higher-dimensional Einstein gravity described by the action

$$S = \int \sqrt{|\underline{g}|}\underline{R} \tag{2.55}$$

In paper V the most general split of this curvature scalar through an APS was derived. As is no surprise from earlier discussions the field content of this split is, beside the two internal curvatures of the complementary distributions, the deformation tensors. Put into their irreducible forms the action reads

$$S = \int \sqrt{g}\sqrt{g'} \left(\tilde{\tilde{R}} + \tilde{\tilde{R}}' + \frac{1-k}{k}\kappa^2 + \frac{1-k'}{k'}\kappa'^2 - 2\underline{\nabla} \cdot \kappa_I + W^2 + W'^2 + L^2 + L'^2 \right) \tag{2.56}$$

Although this has generally nothing to do with strings this will be referred to as the string frame. One could incorporate the Einstein frame in which the dimensionally reduced curvature term would simply be the Einstein term, i.e. $\sqrt{|\underline{g}|}\underline{R} \to \sqrt{|g|}\tilde{\tilde{\underline{R}}}$. To obtain this one has to make a conformal transformation of the metric, but for the purpose of studying the geometry of the theory the string frame is the more natural to choose. As is seen from the above action, beside the deformation tensor parts there is also a total derivative which will come into play when studying manifolds with boundary. To identify the different theories through the APS theory, one first notices, that from paper IV, the twisting tensor is simply $L = \frac{1}{2}F$, where F is the gauge field strength. Ordinary gauge theory is found to be of the type (GF, GD) and thus by taking all fields but L to zero the action reduces to

$$S = \int \sqrt{g}\sqrt{g'} \left(\tilde{\tilde{R}} + \tilde{\tilde{R}}' + L^2 \right) \tag{2.57}$$

Taking the base manifold to be ordinary Minkowski space and dropping the constant normal curvature term plus rewriting the action into the Einstein frame this can be written

$$S = \int \sqrt{g}\frac{1}{4g^2}F^2 \tag{2.58}$$

By this scheme the coupling constant is identified with some power of the internal radius of the gauge group, i.e. $g = R^\beta$ where β is some constant. The main difference of Kaluza–Klein theory in comparison with ordinary gauge theory is that this radius often parameterized with the dilaton field is taken to be a dynamical object. Here the κ'^2 term contains the dynamics of the dilaton and the structure will basically be (UF, GD). In order to keep all scalar degrees of freedom upon compactification one would end up with a (F, GD) structure. The total action describing this reads

$$S = \int \sqrt{g}\sqrt{g'}\left(\tilde{\tilde{R}} + \tilde{\tilde{R}}' + \frac{1-k'}{k'}\kappa'^2 - 2\underline{\nabla}\cdot\kappa' + W'^2 + L^2\right) \tag{2.59}$$

Here the conformation tensor consists of the other scalar degrees of freedom which not serves as overall conformal factors but which instead represents some internal volume preserving deformations. As is clear from the above analysis the general case looks a lot richer, but by only calculating the degrees of freedom it looks like as the Kaluza–Klein ansatz would contain all possible information. This is not true, though, as was discussed when conjecturing new topological non-trivial solutions with non-vanishing normal twisting. Locally the metric can be put into such form that the Kaluza–Klein ansatz looks general. The κ and W fields would here only represent higher modes in mass parameters associated with the normal coordinates. A more interesting thing noticed through the above analysis is that all these theories are found in a bigger theory with only one dynamical field, the metric. Dimensionally this implies that that there are only two measurable quantities namely length and time. Time is of course implicit in the metric structure of the locally minkowski space. This on the other hand implies that there are only two natural physical constants and not three as is usually referred to. These are of course the Planck length and the speed of light. What is striking here is that it is the gravitational coupling constant that is superfluous and can be included in the other fields. The planck constant is replaced with the planck length through the normal relation. All electric an magnetic charges will all have the dimension of length to some power and the Dirac quantization conditions will be purely geometrical. Interesting to note is that charged spin 1/2 particles in four dimensions obey the quantization condition

$$\frac{q_e q_m}{4\pi} = \frac{1}{2} l_p^2 \tag{2.60}$$

In these geometrical units mass will be of dimension length and the reciprocal length usually associated with the mass is simply m/l_p^2.

Duality conjectures including the interchange of the coupling constant with its inverse, i.e. $g \to \frac{1}{g}$, will in this geometrical picture be identified with geometries preserving the structure when transforming $R \to \frac{1}{R}$ for some internal degree of freedom. So geometrically S-duality and T-duality looks pretty much the same with the winding of the string replaced by the winding of some internal $U(1)$. This will be looked upon in the sequel.

3

Solitons

The first notion of solitons was due to the Scottish engineer John Scott Russell (1808-1882) when he in 1834 discovered the "solitary wave". What he saw was a different type of water wave appearing in the domain of a canal. It was a wave that did not dissipate but remained in size and shape as it wandered down the stream. Although J.S. Russell never succeeded in proving that these solitons actually were a class of solutions to the hydro-dynamical equations of motion, he never doubted its existence. It was not until after his death in 1895 that Korteweg and de Vries should give a complete analytical explanation, known as the *soliton* solution to the nonlinear hydrodynamical equation, known as the Korteweg–de Vries equation. These types of solutions only arise in certain classes of non-linear differential equations, of which some examples will be given in this chapter as they appear widely in interacting field theory.

Field theory can in general be divided into a non-interacting and an interacting part. The non-interacting part contains the kinetic term and is alone called a free field theory. The interacting part is usually equipped with some coupling constant whose value determines the strength of the interaction. Conventionally the coupling constant is chosen such that as if it is set to zero the interaction vanishes. In quantum field theory one usually regards the coupling constant as small and solves the equations of motions for the free field theory with oscillators interpreted as creation and annihilation operators, and does perturbation theory in terms of this small coupling constant. These terms are interpreted with Feynman diagrams as some sequence of particle creations and annihilations coupled to the otherwise free propagating particle. Now what makes solitons extra interesting is that they are complete solutions to the non-linear equations of motions and, as it turns out, their mass is inversely proportional to some power of the coupling constant. The direct conclusion made from this is that they are not seen in conventional quantum field theory. We say that they lie in the non-perturbative spectrum of the theory. Before giving some examples it would be interesting to state all typical characteristics of soliton solutions to non-linear field theory.

- Solitons are localized classical solutions to a non-linear field theory which are finite in energy.
- Solitons are non-perturbative in the sense that their mass is inversely proportional to some power of a dimensionless coupling constant.
- Solitons are stable solutions which are characterized by a topological charge rather than a Noether charge.
- Solitons can be regarded as an interpolation between topologically inequivalent vacua.
- Solitons are classical solutions that depend on a finite number of parameters which are called moduli. These parameters can be seen as coordinates on the moduli space which differs for different topological charge.

Interesting to note is that although the soliton is localized in space, it is not a point-like object but rather a "lump" of energy. The reason for the existence of solitonic objects can be traced down to a non-trivial vacuum structure associated with the spontaneous breaking of some internal symmetry of the field theory in question. The solutions can be put into different topological sectors characterized by the winding of a mapping from the boundary of the space to the vacuum manifold. We will give three similar examples of this all characterized by the Higg's mechanism. This mechanism is basically built upon a gauge valued scalar field with a potential term which goes as the 4:th power in the Higgs field. The reason for this is of course to give us the non-degenerate vacuum necessary for the spontaneous symmetry breaking. It should be stressed that the ordinary index conventions used in this thesis will be sidestepped in this chapter for convenience.

3.1 1+1 Domain walls

Let us start with 1+1-dimensinal ϕ^4-theory, described by the action

$$S[\phi] = \int d^2x \left(-\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4}(\phi^2 - \phi_0^2)^2 \right), \tag{3.1}$$

where $\phi_0^2 = m^2/\lambda$ is constant. This theory has a $O(1) = \mathbb{Z}_2$ symmetry by reflection, i.e. $\phi \to -\phi$ which implies that there are two different vacuum solutions, namely

$$\phi = \pm\phi_0. \tag{3.2}$$

We say that the vacuum manifold, $\mathcal{V}$, is two-fold degenerate, i.e. $\mathcal{V} \cong \mathbb{Z}_2$. In quantum field theory one chooses one specific vacuum around which perturbations are done. The perturbative spectrum of this two-dimensional theory consists of meson excitations of mass m which then describe fluctuations around any of these vacua. But there is also the non-perturbative excitation in term of the soliton solution which is an interpolation between these two inequivalent vacua, i.e. with the property

$$\phi(x) \to \begin{cases} \phi_0 & x \to \infty, \\ -\phi_0 & x \to -\infty. \end{cases} \tag{3.3}$$

But forcing the soliton to interpolate between two vacua costs energy so the soliton acquires mass. A trick due to Bogomol'nyi gives us the possibility of rewriting the energy-density of a static solution, $\mathscr{E}$, in the form

$$\mathscr{E} = \frac{1}{2}\left(\frac{d\phi}{dx} + \sqrt{\frac{\lambda}{2}}(\phi^2 - \phi_0^2)\right)^2 - \sqrt{\frac{\lambda}{2}}\frac{d}{dx}\left(\frac{\phi^3}{3} - \phi_0^2\phi\right). \tag{3.4}$$

As is seen the last term is a total derivative so it is a boundary term which is nothing but a multiple of the topological charge associated with this soliton solution. By first introducing a dimensionless coupling constant $g^2 := \lambda/m^2$ and the topological current

$$J^\mu = \frac{1}{2\phi_0}\epsilon^{\mu\nu}\partial_\nu\phi, \tag{3.5}$$

which is automatically conserved, i.e., $\partial_\mu J^\mu = 0$, without the use of equations of motion. The topological charge associated with this current is

$$Q = \int_{-\infty}^{+\infty} J^0 dx = \frac{1}{2\phi_0}(\phi(+\infty) - \phi(-\infty)) = \begin{cases} 1 & \text{Kink,} \\ 0 & \text{Vaccum,} \\ -1 & \text{Anti-Kink.} \end{cases} \tag{3.6}$$

The 'Kink' solution is the one associated with the boundary conditions given in eq. 3.3. All these solutions are of course stable since the topological charge, Q, is a constant of motion. Now equation 3.4 gives a lower bound for the mass of the soliton known as the *Bogomol'nyi bound.*

$$M \geq \frac{2\sqrt{2}}{3}\frac{m}{g^2} \mid Q \mid. \tag{3.7}$$

When this bound is saturated the soliton is called a *BPS-state*. BPS-states have more remarkable effects when studying supersymmetric field theories as they correspond to different representations, known as *short multiplets*, contrary to non-BPS configurations. In this simple case one can find the solution which *saturates* the Bogomol'nyi bound by minimizing the energy given in eq. 3.4. This is done by solving the first order differential equation

$$\frac{d\phi}{dx} + \sqrt{\frac{\lambda}{2}}(\phi^2 - \phi_0^2) = 0. \tag{3.8}$$

The solutions to this differential equation is nothing but the 'Kink' and 'Anti-Kink' solitons given by

$$\phi = \pm\phi_0 \tanh\left(\frac{m}{\sqrt{2}}(x - x_0)\right). \tag{3.9}$$

Here x_0 is a constant of integration and is a free parameter of the solution. It can be regarded as the position of the soliton. This parameter can be regarded as the modulus of the 'Kink' or 'Anti-Kink', so we see that the moduli space of this solution is $\mathcal{M} \cong \mathbb{R}$. Note that the solutions approach the asymptotic solutions $\phi(\pm\infty) = \pm\phi_0$, as they should. As they now saturate the Bogomol'nyi bound, their masses are given by

$$M = \frac{2\sqrt{2}}{3}\frac{m}{g^2} \mid Q \mid . \tag{3.10}$$

From this mass relation it is clear that the weaker the coupling the heavier the soliton or as $g \to 0$, the soliton lies in the non-perturbative spectrum of the theory. In strong coupling on the other hand the soliton become light and will dominate the dynamics of the theory. In these days there is a lot of interest in the concept of duality in which a theory is invariant under the exchange of solitons with fundamental particles when one at the same time changes the coupling constant, $g \to 1/g$. This is of the type strong-weak duality also known as S-duality but we will leave the discussion of this until chapter 6 where we look at dualities in string theory.

3.2 1+2 Vortices and Strings

Here will be considered an $O(2)$ theory, but as $O(2)$ basically is the same as $U(1)$, the theory can be rewritten into equivalent $U(1)$ form instead. As for the domain wall the presence of a string only arises through dimensional oxidation to $1+3$–dimensions. The Lagrangian describing the dynamics of the theory is given by

$$\mathcal{L} = -\frac{1}{4}F^2 - |D\phi|^2 - \mu^2|\phi|^2 - \lambda|\phi|^4, \tag{3.11}$$

where $F = dA$ is the abelian gauge field strength with respect to the local $U(1)$ gauge invariance and ϕ is a complex scalar. This theory has a degenerate vacuum when $\mu^2 < 0$ given by

$$\phi(x) = e^{i\alpha(x)}\phi_0, \qquad A_i(x) = \partial_i\alpha(x), \qquad \phi_0 = -\frac{\mu^2}{2\lambda}. \tag{3.12}$$

Suppose that the Higgs field is everywhere non-vanishing, then the vacuum can, by a local gauge transformation, be reduced to $\phi(x) = \phi_0, A_i(x) = 0$. Doing perturbations around this vacuum this spontaneous symmetry breaking will give both the Higgs field and the vector boson (the A–field) mass given by

$$m_H^2 = 4\lambda\phi_0^2, \qquad m_V^2 = 2e^2\phi_0^2, \tag{3.13}$$

where e^2 is the coupling constant of the abelian gauge field. If the Higgs field on the other hand vanishes for some point (or points) the direction (gauge) is not defined in that point. This gives the possibility for the Higgs field to have a topologically non-trivial winding. As $r \to \infty$ the Higgs field must reach its vacuum expectation value but this can now be written (up to local gauge transformations) like

$$\phi \to \phi_0 e^{in\theta}, \quad \text{as} \quad r \to \infty, \tag{3.14}$$

where r, θ are plane polar coordinates. As the requirement that physical excitations have finite energy one can look at the static energy and see what other restrictions this put on the fields.

$$\mathcal{E} = \frac{1}{4} F_{ij}^2 + |D_i \phi|^2 + \lambda (|\phi|^2 - \phi_0^2)^2 \tag{3.15}$$

Here it is obvious that for the energy to be finite one have the additional requirement

$$D_i \phi(x) \to 0 \quad \text{as} \quad r \to \infty \tag{3.16}$$

This condition can be used to solve for the gauge field, which then reads

$$A_i \to \frac{n}{e} \partial_i \theta(x, y) \quad \text{as} \quad r \to \infty. \tag{3.17}$$

So the gauge field tends to a *pure gauge* at the boundary which implies that the gauge field strength tends to zero and enables the possibility of a finite energy solution. Another most intriguing implication of this is that the topologically non-trivial boundary condition of the Higgs field leads to a magnetic charge which is quantized in terms of the electric coupling constant, i.e.

$$q_m = \int F = \oint_{S^1_\infty} A_i dx^i = \frac{2\pi n}{e} \tag{3.18}$$

This is nothing but a *Dirac quantization condition* which looks more manifest if one identifies the smallest electric charge, q_e, by $q_e = e$. The quantization condition then reads

$$q_e q_m = 2\pi n. \tag{3.19}$$

So we have seen for this 2+1–dimensional theory with a $U(1)$ gauge symmetry contains topologically inequivalent vacua characterized by the winding of the Higgs field at the boundary. The vacuum manifold is given by $\mathcal{V} \cong G/H = U(1) \cong S^1$ and the Higgs field defines a map

$$\frac{\phi}{|\phi|}: \quad S^1_\infty \longmapsto G/H \cong S^1 \tag{3.20}$$

$$x \longmapsto g(x) \in G/H \tag{3.21}$$

The topological current defined by this map is given by

$$J^\mu = -\frac{i}{2\pi\phi_0^2}\epsilon^{\mu\nu\rho}\partial_\nu\bar{\phi}\partial_\rho\phi \tag{3.22}$$

and the corresponding topological charge reads

$$Q = \int d^2x J^0 = n \tag{3.23}$$

Until now the discussion has just regarded properties of the solitons in case they exist but there has not been a general proof of their existence. It turns out though that the equations of motions are not possible to solve analytically but it is quite easy to show their existence or to solve them numerically. Without doing any of these alternatives we can nevertheless tell additional properties by doing the Bogoml'nyi trick to rewrite the energy density in the form

$$\begin{aligned}\mathcal{E} &= |(D_1 + iD_2)\phi|^2 + \frac{1}{2}[F_{12} + e(|\phi|^2 - \phi_0^2)]^2 \\ &\quad + e\phi_0^2 F_{12} + (\lambda - \frac{e^2}{2})(|\phi|^2 - \phi_0^2)^2 - i\epsilon^{ij}\partial_i(\bar{\phi}D_j\phi)\end{aligned}$$

The last term vanishes upon integration since it is a total derivative and $D_i\phi \to 0, r \to \infty$, and if $\lambda \geq e^2/2$ there is a lower bound for the mass

$$M = \int d^2x\mathcal{E} \geq 2\pi|Q|\phi_0^2 = \frac{\pi m_V^2}{e^2}|Q| \tag{3.24}$$

The case $\lambda = e^2/2$ is special because then there exists BPS states. In this case the Bogomol'nyi trick reduces the field equations to first order but neither in this case they are analytically solvable. In this case it is clear that the Higgs field and the vector boson have equal masses. The mass for the 1-vortex is given by

$$M_1 = \frac{\pi m_V^2}{e^2} \tag{3.25}$$

and it is clear that it is non-perturbative in the coupling constant e^2. Further it must be stressed that the solitonic solutions also in this case have integration constants associated with them. These are the so called moduli parameters and as in the domain wall solution associated with the location of the Vortex. The moduli space is therefore $\mathcal{M}_1 \cong \mathbb{R}^2$ for the 1-Vortex. Interesting to note about this model is that it plays an important role in both *Grand Unified Theories* and in non-relativistic superconductors. In GUT it describes a cosmic string upon dimensional oxidation with string tension $(2\pi\alpha)^{-1} = 2\pi\phi_0^2$ and in superconductivity it is precisely the effective Landau-Ginzburg theory. Here the value $\lambda = e^2/2$ corresponds to the border between type I and type II superconductors. In type II superconductors there exist magnetic flux tubes corresponding to the vortices discussed above but in type I superconductors they do not exist.

3.3 1+3 't Hooft–Polyakov monopoles

In this section we will describe the solitonic monopole solutions to the Georgi-Glashow model. These were first found by 't Hooft [14] and Polyakov [15]. The Georgi-Glashow model is an $SO(3)$ (or $SU(2)$) gauge theory coupled to a 'isovector' Higgs field, ϕ^a. That is to say that it transform under the **3** of $SO(3)$. [1] The model is described by the following Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a - \frac{1}{2}(D_\mu \phi)_a (D^\mu \phi)^a - \frac{\lambda}{4}\left(\phi^2 - \phi_0^2\right)^2 \tag{3.26}$$

where the covariant derivative of the scalar Higgs field is given by

$$(D_\mu \phi)^a = \partial_\mu \phi^a - e\epsilon^a{}_{bc} V_\mu^b \phi^c \tag{3.27}$$

and

$$F_{\mu\nu}^a = \partial_\mu V_\nu^a - \partial_\nu V_\mu^a - e\epsilon^a{}_{bc} V_\mu^b V_\nu^c \tag{3.28}$$

ϵ_{abc} being the Levi–Civita of $SO(3)$. The potential term is on the same form as in previous examples and gives rise to spontaneous symmetry breaking when taking a vacuum solution into account. We will see that also in this case it is the vacuum expectation value of the Higgs field that does the trick when it comes to the various monopole configurations. But let us start by giving the spectrum of the model under the symmetry breaking due to choosing a vacuum, say

$$\phi^a = \phi_0 \delta_3^a \qquad\qquad V_\mu^a = 0 \tag{3.29}$$

As is clear from the total energy of any field configuration given by

$$E \equiv \int d^3x\ \theta_{00} = \int d^3x \left\{ \frac{1}{2}\left[(B_i^a)^2 + (E_i^a)^2 + (\Pi^a)^2 + \left[(D_i\phi)^a\right]^2 \right] + V(\phi) \right\} \tag{3.30}$$

where

$$\Pi_a = \left(D^0 \phi\right)_a \qquad F_a^{i0} = E_a^i \qquad F_{a\ ij} = -\epsilon_{ijk} B_a^k \tag{3.31}$$

this is a true vacuum in the sense that the energy is zero which will not be true for the monopole configurations. The vacuum breaks the $SO(3)$ symmetry of the Lagrangian down to the stability group $SO(2) \cong U(1)$ and at the same time the gauge field generators of the broken symmetries

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(V_\mu^1 \pm i V_\mu^2\right) \tag{3.32}$$

[1] In the case of $SU(2) \cong Spin(3)$ we could consider a 'isospinor' Higgs field in the **2** of $SU(2)$. This Higgs filed would than carry half integer electric charge under the remaining $U(1)$ gauge symmetry after spontaneous symmetry breaking. This would alter the outcome of the Dirac quantization condition to be derived.

acquires mass

$$M_W = \phi_0 |q| = \phi_0 e \tag{3.33}$$

where q is their electric charge under the unbroken $U(1)$. Its generator Q_e is given by the generator of the $SU(2)$ gauge group that leaves invariant the VEV of the Higgs field. One can use the Higgs field as a projector along itself to get the associated generator

$$Q_e = e \frac{\phi^a}{\phi_0} T_a = e T_3 \tag{3.34}$$

Also the Higgs field acquires a mass under the symmetry breaking

$$M_H = \sqrt{2\lambda}\ \phi_0 \tag{3.35}$$

The vacuum manifold is given by $\mathcal{V} = G/H = SO(3)/SO(2) = S^2$. We could now search for VEV for the Higgs field with winding analogous to the case of the Vortex. The simplest boundary condition would be

$$\phi^a \to n^a(x^i)\phi_0, \quad r \to \infty \tag{3.36}$$

where n^a is an outward pointed normal unit vector given by

$$n^a(x^i) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \frac{x^a}{r} \tag{3.37}$$

but we could of course have an arbitrary winding which can be expressed like

$$n^a(x^i) = (\sin\theta\cos k\varphi, \sin\theta\sin k\varphi, \cos\theta) \tag{3.38}$$

instead. Now it is clear that the Higgs field again describes a mapping from the boundary sphere at infinity to the sphere of the vacuum manifold, i.e.

$$\frac{\phi}{|\phi|}: \quad S^2_\infty \longmapsto G/H \cong S^2 \tag{3.39}$$

$$x \longmapsto g(x) \in G/H \tag{3.40}$$

The integer k in 3.38 denotes the winding number of this map. Again we have a topological current

$$J_\mu = \frac{1}{8\pi\phi_0^3} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abc} \partial_\nu \phi^a \partial_\rho \phi^b \partial_\sigma \phi^c \tag{3.41}$$

which topological charge is given by the winding number k, i.e.

$$Q = \int d^3x \ J^0 = k \tag{3.42}$$

It will turn out that this topological charge will give rise to a magnetically charged monopole under the unbroken gauge group $U(1)$. It will also give a Dirac-like quantization condition for magnetically charged objects and electrically charged ones. To find out this we first note that again finite energy requires the covariant derivative of the Higgs field to turn to zero at infinity, i.e.

$$D_\mu \phi = 0 \qquad\qquad \phi^2 = \phi_0^2 \tag{3.43}$$

This condition can be solved for the gauge field which then reads

$$V_\mu^a = \frac{1}{\phi_0^2 e} \epsilon^a{}_{bc} \phi^b \partial_\mu \phi^c + \frac{\phi^a}{\phi_0} A_\mu \tag{3.44}$$

where A_μ is arbitrary. This arbitrary field is nothing but the unbroken $U(1)$ gauge field lying in the direction of the Higgs field. As was said earlier the Higgs field could be regarded as a projector onto the unbroken stability group $U(1)$ of the gauge group $SO(3)$. In last equation it is manifest that the vectorfield A_μ lies in the unbroken $U(1)$ direction while the other term is orthogonal to the Higgs field. From this relation the gauge field strength can be calculated and is seen to point entirely in the direction of the Higgs field.

$$F_{\mu\nu}^a = \frac{\phi^a}{\phi_0} F_{\mu\nu} \tag{3.45}$$

where

$$F_{\mu\nu} = \frac{1}{e\phi_0^3} \epsilon_{abc} \phi^a \partial_\mu \phi^b \partial_\nu \phi^c + \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.46}$$

The first term in the $U(1)$ field strength contain the winding of the Higgs field. In fact from 3.41 it is clear that the field strength satisfies the Maxwell equation with a possible monopole configuration given by the Higgs field, i.e.

$$\partial_\mu F^{\mu\nu} = 0 \qquad\qquad \partial_\mu {}^* F^{\mu\nu} = \frac{4\pi}{e} J^\mu \tag{3.47}$$

The magnetic part of the $U(1)$ field strength is given by

$$B^i = \frac{\phi^a}{\phi_0} B_a^i = \frac{1}{2} \epsilon^{ijk} F_{jk} = \frac{1}{2} \epsilon^{ijk} \frac{1}{e\phi_0^3} \epsilon_{abc} \phi^a \partial_j \phi^b \partial_k \phi^c \tag{3.48}$$

and the magnetic flux is given by

$$q_m = \int_{S_\infty^2} B^i dS_i = \int d^3x \partial_i B^i = \frac{4\pi}{e} \int d^3x J^0 = \frac{4\pi}{e} Q \tag{3.49}$$

So we see that if we take $q_e = e$ to be the fundamental electric charge we get the Dirac quantization condition

$$q_e q_m = 4\pi n \tag{3.50}$$

which have an additional factor two in comparison with the original. This factor two can be removed if one takes into account the spinor representation of $SO(3)$ which is $SU(2)$ and take the fundamentally electrically charged particles to those lying in the fundamental representation of the spin group namely the **2** of $SU(2)$. These will carry charge $q_e = \frac{e}{2}$ and we are back at the original Dirac quantization condition. Now as we saw in the case of the Vortex, imposing boundary conditions on the Higgs field with winding the Higgs field must be zero at some location in the bulk which is regarded as the position of the monopole. But forcing the Higgs field to zero costs energy and the monopole becomes massive. This mass can be given a lower bound by the standard Bogomol'nyi trick.

$$(B_i^a)^2 + [(D_i\phi)^a]^2 = [B_i^a \pm (D_i\phi)^a]^2 \mp 2B_i^a\,(D_i\phi)^a \tag{3.51}$$

The energy can then be written like

$$E = \int d^3x \frac{1}{2}\,[B_i^a \pm (D_i\phi)^a]^2 \mp B_i^a\,(D_i\phi)^a + \frac{\lambda}{4}\left(\phi^2 - \phi_0^2\right)^2 \tag{3.52}$$

which implies that

$$M \geq \mp \int d^3x B_i^a\,(D_i\phi)^a = \frac{4\pi}{e}\phi_0|Q| = 4\pi\frac{M_W}{e^2}|Q| \tag{3.53}$$

The BPS limit can only be obtained if we let $\lambda \to 0$ which leaves us with a 1-monopole mass given by

$$M_1 = 4\pi\frac{M_W}{e^2} \tag{3.54}$$

Again the solitonic monopole turns out to be non-perturbative in the coupling constant.

3.4 Duality in field theory

In some field theories there appears a symmetry which is of a completely different origin compared to ordinary gauge symmetries. In gauge theories or in Einstein's theory of gravity the symmetries are gauge invariance or diffeomorphism invariance of the action describing the dynamics of the respective theory. The new symmetry, referred to as duality, involves the interchange of fields and coupling constants. Basically duality can be described as an interchange between two different field constituents which often are dominant in two different regimes of perturbation theory. There are two basic features which characterize duality, namely

- Duality interchanges fundamental electrically charged particles with magnetically charged solitons.
- Duality reverses the role of strong and weak coupling.

To illustrate the role of duality let us first consider pure electromagnetic field theory and discuss duality from its point of view. Maxwell's equations describing the dynamics of the electric and magnetic fields coupled to an ordinary electric source read

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho & (3.55)\\ \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} & (3.56)\\ \nabla \cdot \mathbf{B} &= 0 & (3.57)\\ \nabla \times \mathbf{B} &= \frac{1}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} & (3.58) \end{aligned}$$

Here the speed of light is inserted to stress that a duality between the electric and magnetic field need not be that obvious after all. As is seen from the Maxwell's equations the coupling between the electric and magnetic fields vanishes if $c = \infty$. In that case the fields are completely independent of each other and one could argue by slowly retard the speed of light, that magnetism is nothing but a drag effect of the inability of the electric field to change over the whole region instantaneously. Nevertheless the appearance of non-unital coupling constants suggests that this entanglement does depend on different factors and that duality should be taken seriously.

Dropping the electrical source term the free Maxwell's equations can quite easily be seen to possess an $SO(2)$ duality symmetry. The free field equations are

$$\begin{array}{ll} \nabla \cdot \mathbf{E} = 0 & \qquad \nabla \times \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0 \\ \nabla \cdot \mathbf{B} = 0 & \qquad \nabla \times \mathbf{B} - \dfrac{\partial \mathbf{E}}{\partial t} = 0 \end{array}$$

Here the speed of light is put to one again. The $SO(2)$ duality transformation leaving the free field equations invariant is simply

$$\mathbf{E} \to \mathbf{E}\cos\phi - \mathbf{B}\sin\phi \qquad (3.59)$$

$$\mathbf{B} \to \mathbf{B}\cos\phi + \mathbf{E}\sin\phi \qquad (3.60)$$

This duality transformation can most elegantly be rewritten in terms of a complex vector field $\mathscr{V} = \mathbf{E} + i\mathbf{B}$ by which the duality transformation becomes

$$\mathscr{V} \to e^{i\phi}\mathscr{V} \qquad (3.61)$$

The energy density and the Poynting vector can in terms of this complex vector field be rewritten

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) = \frac{1}{2}|\mathscr{V}|^2 & (3.62)\\ \mathcal{P} &= \mathbf{E} \times \mathbf{B} = \frac{1}{2i}\mathscr{V}^* \times \mathscr{V} & (3.63) \end{aligned}$$

These quantities are thus manifestly symmetric under the duality transformation in eq. 3.61 which is a $U(1) \cong SO(2)$ transformation. If one wishes to include matter coupled to the electromagnetic field this duality symmetry vanishes, though, unless one includes magnetic charges and currents in addition to the fundamental electric ones. In complex notation these are included through

$$\rho_c = \rho_e + i\rho_m \tag{3.64}$$
$$j_c = j_e + ij_m \tag{3.65}$$

Maxwell's equations coupled to a source term can thus be written in a manifestly symmetric form under this duality transformation.

$$\nabla \cdot \mathscr{V} = \rho_c \tag{3.66}$$
$$\nabla \times \mathscr{V} = i\frac{\partial}{\partial t}\mathscr{V} + ij_c \tag{3.67}$$

The duality symmetry holds if the source terms are transformed under the same transformation as the complex vector field, i.e.,

$$\rho_c \rightarrow e^{i\phi}\rho_c \tag{3.68}$$
$$j_c \rightarrow e^{i\phi}j_c \tag{3.69}$$

In this particular theory in four dimensions both the electric and the magnetic charges are pointlike objects and denoted by q_e and q_m respectively, duality implies the following transformation

$$q_e + iq_m \rightarrow e^{i\phi}(q_e + iq_m) \tag{3.70}$$

In ordinary theory of electromagnetism magnetically charged particles are absent. This is solidly based on the lack of experimental evidence for their existence. As was seen in previous section their experimental absence could be explained by their considerable higher mass compared to electrically charged particles. As was shown by Dirac [16] the existence of both electric and magnetic charges q_e and q_m in a theory leads, upon quantization of the theory, to a certain condition on the charges, called the Dirac quantization condition,

$$q_e q_m = 2\pi n \tag{3.71}$$

where n is an integer. It turns out though that this electromagnetic field theory is not suited to possess this duality in a consistent way. The Dirac quantization condition is not invariant under the $U(1)$ duality transformation for example. It is only invariant under the discrete transformation obtained from eq.(3.70) for $\phi = -\frac{\pi}{2}$. Here the fields transform according to

$$\mathbf{E} \rightarrow \mathbf{B} \qquad\qquad \mathbf{B} \rightarrow -\mathbf{E} \tag{3.72}$$

This transformation can be seen to be generated by a duality matrix, T, acting on the two-vector consisting of the electric and magnetic fields as

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \tag{3.73}$$

This duality matrix satisfies $T^4 = \mathbb{1}$ and thus generates the discrete duality group $\mathbb{Z}_4$. There is however a familiar $\mathbb{Z}_2$ subgroup of this duality group known as the charge conjugation. Applying the discrete duality transformation twice is equal to applying the charge conjugation operator, i.e.

$$(\mathbf{E}, \mathbf{B}) \to (-\mathbf{E}, -\mathbf{B}) \tag{3.74}$$

Since the charge conjugation symmetry of the electromagnetic field theory is widely known one often refer this discrete duality to be a $\mathbb{Z}_2$ symmetry instead. This discrete duality is seen to exchange electrically charged particles with magnetically charged ones, i.e.

$$q_e \to q_m \qquad\qquad q_m \to -q_e \tag{3.75}$$

Although the duality symmetry in electromagnetic field theory still must be stressed not to be really relevant, the argument due to the Dirac quantization condition is not entirely true. The reason is that when including magnetically charged particles into the theory one should rather include dyons which are both electrically and magnetically charged. Taken two such dyons with electric and magnetic charges equal to (q_e^i, q_m^i) and (q_e^j, g_m^j) respectively and enforcing the angular momentum of the electromagnetic field to be half-integer quantized we end up with a generalization of the Dirac quantization condition, namely

$$q_e^i q_m^j - q_e^j q_m^i = 2\pi n^{ij} \tag{3.76}$$

where n^{ij} is an integer. This goes under the name of Dirac–Schwinger–Zwanziger [17, 18] (DSZ) quantization condition. It can be seen to be invariant under the $U(1)$ duality transformation by introducing the complex charge by $q_c := q_e + iq_m$ which makes the DSZ quantization condition to look like

$$\mathrm{Im} q_c^i q_c^{j*} = 2\pi n^{ij} \tag{3.77}$$

This quantization condition is now manifestly duality invariant. Before discussing theories which more realistically possess duality symmetries the problems with the electromagnetic theory which thus far have been seen to work out, must be stressed. It is when taken into account for modern analysis of gauge theories when the duality in the electromagnetic theory fails. Notice that

$$\frac{1}{2}\mathscr{V}^2 = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) + i\mathbf{E}\cdot\mathbf{B} \tag{3.78}$$

which are respectively the Lagrangian of the electromagnetic field and the topological charge density. These are thus not duality invariant but transform as a doublet with an angle equal to 2ϕ under the duality transformation. This means that the Lagrangian is not duality invariant but because of the doublet transformation indeed is invariant under charge conjugation. It is the vector potential needed to derive Maxwell's equations that does not admit the duality transformation. By this date there is no doubt that the physical origin lies in vector potential for several reasons and thus rejects this electromagnetic duality. On the other hand the duality on the field equations level have been very instructive in understanding the principles behind duality symmetries.

Another interesting theory to study from the duality perspective is the Georgi–Glashow model seen in previous subsection. This was described by a Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^2 + \frac{1}{2}(D\phi)^2 - V(\phi) \tag{3.79}$$

where $V(\phi)$ is the potential term for the Higgs field and must be at least fourth order to get the gauge symmetry breaking required for the existence of magnetic monopole solutions. In the Georgi–Glashow model it was taken to be

$$V(\phi) = \frac{1}{4}\lambda(\phi^2 - \phi_0^2)^2 \tag{3.80}$$

The Higgs vacuum admitting these solutions were

$$D_a\phi = 0, \quad V(\phi) = 0 \tag{3.81}$$

This implies that $\phi^2 = \phi_0^2$ and the stable monopole solutions to the equations of motions could be given from the winding of the Higgs field at infinity,

$$q_m = \frac{1}{2q_e\phi_0^2}\int_{S^2_\infty} \epsilon_{abc}\phi\cdot[\partial_b\phi, \partial_c\phi]dS^a \tag{3.82}$$

which fulfilled the Dirac quantization condition

$$q_e q_m = 4\pi n, \quad n \in \mathbb{Z} \tag{3.83}$$

This theory does however admit dyonic solutions beside the pure magnetically charged ones. The Bogomol'nyi bound for these solutions is revised to

$$M^2 \geq \phi_0^2(q_e^2 + q_m^2) \tag{3.84}$$

and the BPS states are those saturating this lower bound. Their masses are thus completely determined by their charges. In the Georgi-Glashow model by letting $\lambda = 0$, or to be precise let λ approach zero as a so called Prasad-Sommerfield limit, all static monopole solutions are in fact BPS states. In the mass formula one finds a striking $\mathbb{Z}_2$ duality by interchanging magnetic and electric states. This $\mathbb{Z}_2$ duality inverts the coupling constant and are thus a strong-weak duality in which the strong coupling limit of the original theory is the weak coupling limit of its dual. This duality was first proposed by Montonen and Olive [19]. A further analysis shows on the other hand that this model must be extended in order to really fulfill this $\mathbb{Z}_2$ duality. Before discussing what the problems are, the attention will first be put to an extension of this duality through the inclusion of a so called Witten term in the Lagrangian. This term is a non-dynamical topological term counting the instanton number and looks like

$$\frac{\theta}{32\pi^2} F_{ab} \cdot F^{ab} \tag{3.85}$$

By introducing a complex coupling constant of the form

$$\tau := \frac{\theta}{2\pi} + i\frac{4\pi}{e^2} \tag{3.86}$$

and a complex field strength

$$\mathcal{G}^i_{ab} := F^i_{ab} + i * F^i_{ab} \tag{3.87}$$

the part of the Lagrangian describing the dynamics of the gauge field can be rewritten

$$-\frac{1}{32\pi} \mathrm{I}m(\tau \mathcal{G}_{ab} \cdot \mathcal{G}^{ab}). \tag{3.88}$$

Through this procedure the $\mathbb{Z}_2$ duality is extended to a larger $SL(2,\mathbb{Z})$ duality. The masses of the BPS states can be written like

$$M^2 = 4\pi\phi_0^2 \mathbf{n}^t \mathbf{M}(\tau)\mathbf{n} \tag{3.89}$$

where $\mathbf{n}^t = (n_e, n_m) \in \mathbb{Z} \times \mathbb{Z}$ and

$$\mathbf{M}(\tau) = \frac{1}{\mathrm{I}m\tau} \begin{pmatrix} 1 & \mathrm{R}e\tau \\ \mathrm{R}e\tau & |\tau|^2 \end{pmatrix} \tag{3.90}$$

It should be stressed that the only stable dyonic states are those where n_e and n_m are relatively prime. This $SL(2,\mathbb{Z})$ duality group includes the original $\mathbb{Z}_2$ duality as a subgroup so it is an extension of the original strong-weak duality conjectured by Montonen and Olive. But although very instructive this theory must be further extended through supersymmetry in order to possess this duality symmetry as the picture above is a bit to naive in its arguments. Basically Montonen and Olive argued for this duality through two points by which the mass invariance under the duality transformation was the first. Secondly the attention is put to the spectrum which in the BPS limit contains one massless photon, one massless Higgs particle, a massive electrically charged W boson, and a massive magnetic monopole. The stability of the two-monopole solution implies that the force between two equally charged magnetic monopoles vanishes. This is explained by the fact that the force from the coupled Higgs field is equally strong but opposite directed. So the second argument was that if this was the case for the electrically charged W bosons the conjecture suggested a duality through the interchange of W bosons with magnetic monopoles. In the BPS limit this was indeed the case but nevertheless the arguments are not strong enough to really entail this duality to the Georgi–Glashow model. Their are two severe objections to the above duality. For one the W bosons are spin 1 particles while the magnetic monopoles are spin 0 particles. Secondly the mass formula is purely classical and might be revised through quantum corrections. Supersymmetry resolves both these problems. In N=2 Super–Yang–Mills theory Witten and Olive [20] proved that the above mass formula is a direct consequence of the supersymmetry algebra, where both the electric and magnetic charges are parts of the central charge. The preservation of the degrees of freedom through the Higgs mechanism forces the states that acquires mass to be BPS states and the above mass formula is obtained. See next chapter for details. Even so, the N=2 SYM theory is not symmetric enough to contain this duality symmetry. The magnetic monopole does not lie in the same multiplet as the W boson, in this theory which makes duality impossible. The N=4 SYM theory, on the other hand, contains the W boson and the magnetic monopole in the same multiplet and has been shown to satisfy this $Sl(2,\mathbb{Z})$ at the quantum level [21, 22].

Geometrically it was discussed in the last chapter that the dualities exchanging the coupling constant with its inverse could be seen as interchanging the radius of the gauge group by its inverse. The magnetic monopoles which are solitons can be regarded as a winding of the gauge group over the base manifold. When discussing string theory their appears a new type of duality called T-duality where the soliton is changed to a string winding around a compact dimension. More on this in the sequel.

4

Supersymmetry

In the previous chapters gauge theories have been studied from a geometrical point of view. In the context of almost product structures it was clear that Yang–Mills theory and Kaluza–Klein theory were two flavors of the same fruit. Although giving a beautiful geometrical description of the different fields appearing in these theories the APS approach tells us nothing about the fermionic spectrum of the physical theories. As is widely observed in physical experiments there are several particles obeying different statistics to the so called exchange particles appearing in pure Yang–Mills theory and inherently in the APS approach. These are the fermions which are seen to transform under the spinor representation of the orthogonal group. The fermions' dynamics are known to be described by the Dirac equation. To connect a fermion to an exchange particle is done by letting the fermion be gauge valued. It is then connected through what is called the minimal coupling scheme, which is nothing but taking the gauge covariant derivative. When trying to find a unified theory of all interactions or a TOE (Theory Of Everything) this coupling between fermions and bosons seems to be too weak. One would like some connection that picked out the fermion spectrum given the boson spectrum or vice versa. There are some important points that a final TOE must fulfill

- Contain and explain all types of interactions.
- Derive the spectrum and all mass relations.
- It must be built from a couple of fundamental postulates. Or to put it transparently, answer the question why it looks the way it looks and not in another fashion.

These necessary conditions may seem quite obvious but if one takes a look at the theories at hand no one fulfills all these criteria. Yang-Mills theory for example, can never be fundamental because nothing in the theory will choose what particular gauge group should be used. General relativity can not answer what dimension we are supposed to live in. The theory with minimally coupled fermions does not tell why there should be 3 families. Even if the standard model is enlarged into a GUT (Grand Unified Theory) where the gauge group $SU(3) \times SU(2) \times U(1)$ is embedded into an $SU(5)$ or an $SO(10)$ gauge group, which could then explain the coupling angles in the $SU(3) \times SU(2) \times U(1)$, it would still not pick out that peculiar group. It could neither tell how many families of fermions there should be. The Kaluza-Klein project, which seems to be the best attempt in unifying gravity and Yang–Mills theory is not sufficient to be promoted to a TOE. The question of what gauge group has only been reformulated as the question of what dimension and topology the internal space should have. A new attempt to bring light to the situation is the incorporation of supersymmetry. This chapter will contain a brief review of supersymmetry with applications, starting with section 1 containing the basic ideas behind supersymmetry while section 2 will contain a deeper insight in the representations of the supersymmetry algebra in various dimensions. Section 3 will contain supersymmetric extensions of Yang–Mills theory and discuss some of their properties. In section four we will discuss the extension of the Kaluza-Klein project in generalizing gravity to supergravity.

4.1 Symmetry between bosons and fermions

In ordinary field theory the exchange particles of the internal symmetry group are coupled to the matter constituents through the minimal coupling scheme. This does not provide any restrictions to the matter content of the theory which would be demanded in a unified theory. A desirable theory would therefore include some other symmetry with the feature of bringing a stronger connection between the matter constituents and the exchange particles. Classical field theory is based on two different kinds of fields namely fermionic and bosonic fields. They obey different statistics and are by the spin-statistic theorem forced to be half-integer spin and integer spin fields respectively. As the matter constituents are based on fermionc fields while the exchange particles are described by bosonic fields the desired symmetry would be one which is a symmetry between bosons and fermions. This symmetry has been dubbed supersymmetry. Originally, supersymmetry was proposed by Gol'fand and Likhtman in 1971 [23] where they enlarged the Poincaré algebra by fermionic generators, Q, obeying the anti-commutation rule

$$\{Q, Q\} = P \tag{4.1}$$

and

$$[Q, P] = 0 \tag{4.2}$$

There is a classic theorem by Coleman and Mandula [24] which states that there is no way to unify space-time symmetry and the internal gauge symmetry in any other fashion than the trivial direct product. This tells us that the internal gauge group and the space-time do not talk to each other in the sense that eigenvalues of the mass and spin operators cannot depend on the eigenvalues of the internal charge operators. In other words there are no relations between masses and charges within the theory which makes it unable to describe physics in a unified way. Of course, one can argue, in a modern fashion regarding principal bundles that this statement says nothing more than the trivial theorem stating that every principle bundle of a gauge group G over a topologically trivial manifold M must be topologically that of the direct product $M \times G$. In Kaluza–Klein theories with a topologically trivial base manifold there is a dependence between the charges and the masses though. This originates from the additional input concerning the radius of the gauge group which is absent in usual Yang-Mills theory. But although bringing a possible relation between masses and charges, Kaluza–Klein theory gives no input of the matter constituents of the theory. These must still be put in by hand. In a supersymmetric theory all particles have their supersymmetric partner. This ensures the existence of the same number of bosonic and fermionic degrees of freedom which thus brings a connection between matter constituents and exchange particles of the desired kind. From the supersymmetric extension to the Poincaré algebra above it is clear that the generators of the supersymmetry, the Q's, are fermionic. In [25] it was shown they were not just fermionic but merely of the precise spin $\frac{1}{2}$. There are other interesting things that can be read off immediately from the supersymmetry algebra. The fact that Q commutes with P for instance tells that, although relating states of different spins, it commutes with the mass operator and can therefore not give any information of the mass. This implies that all states in a supermultiplet will have the same mass. By incorporating central charges into the algebra there appears a lower bound to this mass in terms of the charges. (These features will be studied more thoroughly in the upcoming section where the representations of the algebra will be presented.) The positivity of the mass operator implies that there are no negative energy eigenvalues in the spectrum, i.e., a supersymmetric theory can not contain any tachyons. Noting that $E \sim \{Q, Q^\dagger\}$ one finds

$$0 \leq < |E| > \sim < \{Q, Q^\dagger\} > = |(Q^\dagger| >)|^2 + |(Q| >)|^2 \tag{4.3}$$

For the vacuum state with zero energy this implies the equivalence

$$E|0> = 0 \quad \Leftrightarrow \quad Q|0> = 0,\ Q^\dagger|0> = 0,\ \forall Q \tag{4.4}$$

which states that supersymmetry is broken if and only if the vacuum energy is higher than zero. So although no observations of supersymmetry has been done through experiments, ones belief is that it is broken by a mechanism similar to the Higgs mechanism.

4.2 Representations of SUSY

Physical systems are in general described by a Lagrangian or an S-matrix which possesses some symmetries. These symmetries form groups and we say that the system is invariant under these specific symmetry groups. The standard example is the Poincaré group which is the symmetry group of special relativity. Most commonly these groups are Lie groups and they can be seen to be generated by their associated Lie algebra. Physical states can be described by state vectors in some representation of these algebras. So if one wants to classify all possible configuration one would most eagerly search for the possible representation of the specific algebra at hand. The representations of the Poincaré algebra for instance was classified by Wigner already in 1946 [26]. Here the result includes the ordinary massive and massless representations characterized by quantum numbers m and s where m is a positive real number and $s = 0, \frac{1}{2}, 1, \ldots$. These quantum numbers are indirectly eigenvalues of the Casimir operators P^2 and W^2 with eigenvalues $-m^2$ and $m^2 s(s+1)$ respectively. Beside these representations, which are commonly referred to as the physical representations of the Poincaré algebra, there are yet the so called tachyonic representations with negative m^2. Here the associated Hilbert space is no longer semi-definite but are plagued with states with negative norm. There are also massless representations with continuous helicity eigenvalues, i.e., s is now a real parameter. These states are usually referred to as unphysical. A system plagued with these kind of unphysical representations is not desirable if one looks at the ultimate goal of a complete unified theory. In this typical example we can conclude that the postulates of special relativity are not strong enough to rule out these representations and therefore not strong enough to exclude them from our physical theory. If the Poincaré algebra is extended to the super-Poincaré algebra, though, these tachyonic representations immediately fall out and due to the step operators in the spin, the Q's, the massless representations with continuous s are also ruled out. This makes the supersymmetric algebra more appealing as it rules out these kinds of unphysical states. Another advantage of the super-Poincaré algebra is that the field representations thereof are very restrictive, contrary to the representations of the Poincaré algebra itself where in principle all kinds of field configurations are possible. In upcoming sections some interesting field representations of the super-Poincaré algebra are dealt with. These include Super–Yang–Mills and Supergravity theories. In next subsections the origin of these representations are dealt with, making a distinction between three different kinds of representation. Here is of course the massless representations and the massive representations. But in contrary to the plain Poincaré case the massive representations must be divided into two different sectors depending on whether there are central charges or not in the algebra.

Dimension	Spinor	Type of spinor	Number of susy
11	32	Majorana	1
10	16	Maj & Weyl	1,2
9	16	Majorana	1,2
8	16	Weyl	1,2
7	16	Dirac	1,2
6	8	Weyl	1, . . . ,4
5	8	Dirac	1, . . . ,4
4	4	Maj or Weyl	1, . . . ,8
3	2	Majorana	1, . . . ,16
2	1	Maj & Weyl	1, . . . ,32

Table 4.1: Minimal spinor representations in various dimensions

4.2.1 Massless representations

All interesting massless field representations of the supersymmetry extended Poincaré algebras can be found in a detailed review by Strathdee [27]. The most interesting representations in string theory and supergravity will be presented here. To begin with we need to classify the different spinor representations in different dimensions. From Wetterich [28] the minimal spinor representations are given in terms of Dirac, Majorana, Weyl and Majorana-Weyl spinors (see Table 4.1). These stand for plain, real, chiral and chiral plus real spinor representations and are most frequently appearing in the physical literature. There is, though, a more suitable classification of the spinors in order to study the supersymmetry representations. Start with the complete algebra including the generators of the automorphism group.

$$
\begin{aligned}
[M, M] &= M \\
[P, M] &= P \\
[P, P] &= 0 \\
[Q, M] &= Q \\
[Q, P] &= 0 \\
\{Q, Q\} &= P + Z \\
[I, I] &= I \\
[I, M] &= 0 \\
[I, P] &= P \\
[I, Q] &= Q \\
[I, Z] &= Z \\
[Z, M] &= [Z, P] = [Z, Q] = 0
\end{aligned}
\tag{4.5}
$$

Here M stands for Lorentz generators, Z for central charges and I denotes the automorphism generators. To see what automorphism groups there are, the suitable classification is whether the spinor is self-conjugate or pairwise-conjugate under complex conjugation plus if there is a real representation or not. These cases will lead

D mod 8	Spinor type	Automorphism group
0,4	Pairwise conjugate	$SU(N) \times U(1)$
1,3	Real	$SO(N)$
2	Real	$SO(N_+) \times SO(N_-)$
5,7	Pseudoreal	$USp(N)$
6	Pseudoreal	$USp(N_+) \times USp(N_-)$

Table 4.2: Automorphism groups for different types of spinor representations

D mod 8	Representation of $Q_{1/2}$	Real dimension $2n$
0,4	$(2_+^{(D-4)/2}, N)_1 + h.c$	$2^{(D-2)/2}N$
1,3	$(2^{(D-3)/2}, N)_1$	$2^{(D-3)/2}N$
2	$(2_+^{(D-4)/2}, N_+, 1) + (2_-^{(D-4)/2}, 1, N_-)$	$2^{(D-4)/2}(N_+ + N_-)$
5,7	$(2^{(D-3)/2}, N)_1$	$2^{(D-3)/2}N$
6	$(2_+^{(D-4)/2}, N_+, 1) + (2_-^{(D-4)/2}, 1, N_-)$	$2^{(D-4)/2}(N_+ + N_-)$

Table 4.3: Transformations of $Q_{1/2}$ under $SO(D-2) \times Aut$.

to the different automorphism groups $U(N)$, $SO(N)$ or $USp(N)$ according to Table 4.2. When studying light-like representations, which contain no central charges, we introduce light cone variables and observe that the algebra of the supercharges must take the form

$$\begin{aligned} \{Q_{1/2}, Q_{1/2}\} &= P_+ \\ \{Q_{1/2}, Q_{-1/2}\} &= \vec{P} \\ \{Q_{-1/2}, Q_{-1/2}\} &= P_- \end{aligned} \tag{4.6}$$

As the D-momentum should be light-like a conventional frame to choose is

$$P_+ = E, \quad P_- = 0, \quad \vec{P} = 0 \tag{4.7}$$

The $Q_{-1/2}$ is now seen to commute with everything and will thus just generate zero-norm states. They can therefore be discarded upon why the physical information entirely lies in the remaining $Q_{1/2}$ generators and in the isotropic subgroup $SO(1, D-1)$ which for the massless case is $SO(D-2)$ (To be more accurate the largest semi-simple subgroup as the isotropic subgroup in fact is $E_{(D-2)}$). Now when counting the dimensions of the various representations recall that only half of the $Q_{1/2}$'s will become creation operators, and so effectively only $n = \frac{1}{4}MN$ generators will create states. M here is the minimal spinor dimension and N the supersymmetry, why we effectively get the dimension to be 2^n basically. Of course representations can be extended by the direct product of some bosonic representation. The results is listed in table 4.3. We have now the power to classify all representations but we will restrict ourselves to those of physical interest, namely those reducing to fields containing not higher helicity than 2 in 4 dimension and no with a spin 3/2 as highest state as it only couples consistently to the graviton. The representations not listed here can be found in ref. [27]. Table 4.4 contains the chosen representations discussed and table 4.5 contain a list of what the field content of these representations are. In order to see in more detail how the representations are explicitly found, the four dimensional case will be worked through explicitly. We will follow the presentation in [29]. The most general $D = 4$ super-Poincaré algebra is the N-extended supersymmetry algebra which looks like

$$\{Q^I_\alpha, \overline{Q}_{\dot{\beta}J}\} = 2\delta^I_J (\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \tag{4.8}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \tag{4.9}$$

From this algebra one can work out the different representations divided into three sectors. First there is the massless, light-like representations with the central charge put to zero, then there are the massive representations with or without the central charge. These are treated in the next coming two subsections. In the massless case though we have $P^2 = 0$ and can choose a reference frame in which $P_\mu = (E, 0, 0, E)$. According to above discussion the N-extended supersymmetry algebra can be put in the form

$$\{Q^I_\alpha, \overline{Q}_{\dot{\beta}J}\} = \delta^I_J \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix} \tag{4.10}$$

$$\{Q^I_\alpha, Q^J_\beta\} = 0 \tag{4.11}$$

Here we see that the $Q^I_2, \overline{Q}_{\dot{2}J}$ can consistently be discarded when working out the massless representations. This will not be true in the massive case though. One can introduce new creation and annihilation operators by the definition

$$a_I = \frac{1}{2\sqrt{E}} Q^I_1 \qquad a^\dagger_I = \frac{1}{2\sqrt{E}} \overline{Q}_{\dot{1}I} \tag{4.12}$$

The algebra of these operators become an N-dimensional Clifford algebra

$$\{a_I, a^\dagger_J\} = \delta_{IJ} \tag{4.13}$$

$$\{a_I, a_J\} = \{a^\dagger_I, a^\dagger_J\} = 0 \tag{4.14}$$

The representations can be worked out starting from a Clifford vacuum, obeying $a_I \mid \Omega >= 0$, and building up the massless supermultiplet by consequently act with the creation operators on it.

$$\frac{1}{\sqrt{n!}} (a_{I_1})^\dagger (a_{I_2})^\dagger \cdots (a_{I_n})^\dagger \mid \Omega > \tag{4.15}$$

with the number n running from 1 to N. The number of states in this expression is $\begin{pmatrix} N \\ n \end{pmatrix}$ so the total number of states in the massless representation becomes

$$\sum_{n=1}^{N} \begin{pmatrix} N \\ n \end{pmatrix} = 2^N \tag{4.16}$$

of course with a matching number of fermionic and bosonic states. The creation operators $a_I^\dagger$ transform as $(0, \frac{1}{2})$ under the Lorentz group and increases therefore the helicity by one-half. So if the vacuum state $\mid \Omega >$ has helicity λ the maximal helicity state becomes

$$\lambda_{\max} = \lambda + \frac{N}{2} \tag{4.17}$$

See Table 4.6 for a list of the physically interesting representations in 4 dimensions.

4.2.2 Massive representations, $Z = 0$

In the massless case half of the supersymmetry generators did not contribute to the spectrum because they generated zero norm states. In the massive case though there are no such zero norm states so all generators will contribute. The isotropic subgroup is no longer $SO(D-2)$ but now $SO(D-1)$ why massive field representations have dimensions different from those of the light-like ones. The automorphism groups are the same though why the structure is the same as in the massless case up to the change of the little group from $SO(D-2)$ to $SO(D-1)$. This means that representations of the supercharges now are with respect to this new little group. This leads to the fact that the total number of states are squared in comparison to the massless case. That is if the number of states were 2^n in the massless case they are now 2^{2n}. For the complete list of the representations see [27]. As an example we will again work through the 4 dimensional case now with the massive condition that $P^2 = M^2$, but again in the absence of central charges. In this case we can go to the rest frame $P_\mu = (M, 0, 0, 0)$, where the N-extended supersymmetry algebra takes the form

$$\begin{aligned} \{Q_\alpha^I, \overline{Q}_{\dot\beta J}\} &= 2M\delta_J^I\delta_{\alpha\dot\beta} && (4.18)\\ \{Q_\alpha^I, Q_\beta^J\} &= 0 && (4.19)\end{aligned}$$

So by introducing new creation and annihilation operators by

$$a_\alpha^I = \frac{1}{\sqrt{2M}}Q_\alpha^I \qquad (a_\alpha^I)^\dagger = \frac{1}{\sqrt{2M}}\overline{Q}_{\dot\alpha I} \tag{4.20}$$

we end up with a Clifford algebra now with the dimension $2N$ which is double the dimension of the Clifford algebra in the massless case.

$$\begin{aligned} \{a_\alpha^I, (a_\beta^J)^\dagger\} &= \delta_{\alpha\beta}\delta^{IJ} && (4.21)\\ \{a_\alpha^I, a_\beta^J\} &= \{(a_\alpha^I)^\dagger, (a_\beta^J)^\dagger\} = 0 && (4.22)\end{aligned}$$

Starting again from a Clifford vacuum state, obeying $a_\alpha^I \mid \Omega >= 0$, all massive representations can be worked out by acting with the creation operators on the

D	N	*Representation*	M	YM	G
6	$(2,0)$	$2^2 = (2,1;1)+(1,1;2)$	X		
		$(2,1;1)\times 2^2 = (3,1;1)+(1,1;1)+(2,1;2)$	X		
		$(1,2;1)\times 2^2 = (2,2;1)+(1,2;2)$		X	
		$(1,2;3)\times 2^2 = (2,2;3)+(1,2;2)+(1,2;4)$		X	
		$(2,3;1)\times 2^2 = (3,3;1)+(1,3;1)+(2,3;2)$			X
6	$(4,0)$	$(1,3;1)\times 2^4 = (3,3;1)+(1,3;5)+(2,3;4)$			X
6	$(2,2)$	$(2,2;1,1)\times 2^4 = (3,3;1,1)+(3,1;1,1)+(1,3;1,1)+$ $(1,1;1,1)+(2,2;2,2)+(3,2;1,2)+$ $(1,2;1,2)+(2,1;2,1)+(2,3;2,1)$			X
6	$(4,4)$	$2^8 = (3,3;1,1)+(1,3;5,1)+(2,3;4,1)+$ $(3,1;1,5)+(1,1;5,5)+(2,1;4,5)$ $(3,2;1,4)+(1,2;5,4)+(2,2;4,4)$			X
7	2	$2^4 = (5,1)+(1,3)+(4,2)$		X	
		$(5,1)\times 2^4 = (14,1)+(10,1)+(1,1)+$ $(5,3)+(4,2)+(16,2)$			X
9	1	$7\times 2^4 = 27+21+7+1+8+48$			X
9	2	$2^8 = 1_{-4}+7_{-2}+21_{-2}+1_0+7_0+$ $27_0+35_0+7_2+21_2+1_4+$ $8_{-3}+8_{-1}+48_{-1}+48_1+8_1+8_3$			X
10	$(1,0)$	$2^4 = 8_v+8_c$		X	
		$8_v\times 2^4 = 35_v+28_v+1+8_s+56_c$			X
10	$(1,1)$	$2^8 = 35_v+56_v+28_v+8_v+1+$ $8_s+8_c+56_s+56_c$			X
10	$(2,0)$	$2^8 = 1_{-4}+(28_v)_{-2}+(35_v)_0+$ $(35_-)_0+(28_v)_2+1_4+$ $(8_s)_{-3}+(56_s)_{-1}+(56_s)_1+(8_s)_3$			X
11	1	$2^8 = 44+84+128$			X

Table 4.4: Light-like supersymmetry representations

dim	$Scalar$	$Vector$	$2-form$	$3-form$	$4-form$	$Graviton$	$Spinor$	$Gravitino$
D	ϕ	A_m	A_{mn}	A_{mnp}	A_{mnpq}	g_{mn}	χ	ψ_m
4	(0)	(± 1)	(0)	$-$	$-$	(± 2)	$(\pm \frac{1}{2})$	$(\pm \frac{3}{2})$
5	1	3	3	1	$-$	5	2	4
6	$(1,1)$	$(2,2)$	$(3,1),(1,3)$	$(2,2)$	$(1,1)$	$(3,3)$	$(2,1),(1,2)$	$(3,2),(2,3)$
7	1	5	10	10	5	14	4	16
8	1	6	15	$10,\overline{10}$	15	20	$4,\overline{4}$	20
9	1	7	21	35	35	27	8	48
10	1	8	28_v	56_v	$35_+,35_-$	35_v	$8_+,8_-$	$56_+,56_-$
11	1	9	36	84	126	44	16	128

Table 4.5: Field representations in various dimensions under the light-like little group $SO(D-2)$. In $D=4$ the helicities are listed instead.

N	$Representation$	M	YM	G
1	$1_{-1/2}+1_0+1_0+1_{1/2}$	X		
	$1_{-1}+1_{-1/2}+1_{1/2}+1_1$		X	
2	$1_{-1/2}+2_0+1_{1/2}$	X		
	$1_{-1}+2_{-1/2}+1_0+1_0+2_{1/2}+1_1$		X	
4	$1_{-1}+4_{-1/2}+6_0+\overline{4}_{1/2}+1_1$		X	
8	$1_{-2}+8_{-3/2}+28_{-1}+56_{-1/2}+70_0+\overline{56}_{1/2}+\overline{28}_1+\overline{8}_{3/2}+1_2$			X

Table 4.6: Light-like representations for $D=4$

vacuum. The total number of massive states is squared in comparison to the number of massless states and are

$$\sum_{n=1}^{2N} \left(\begin{array}{c} 2N \\ n \end{array} \right) = 2^{2N} \tag{4.23}$$

In the upcoming subsection we will see how the spectrum can be altered by introducing central charges to the algebra. Here the familiar BPS states will reduce the multiplets down to short ones.

4.2.3 Massive representations, $Z \neq 0$

If we add central charges to the algebra the creation operators will turn out to be basically of the form

$$\{a_{\pm}, a_{\pm}^{\dagger}\} = m \pm z \tag{4.24}$$

where z here represents all kinds of central charges. Now from the positivity condition we talked about earlier we get in fact an upper bound for the central charges, namely

$$z \leq m \tag{4.25}$$

This is nothing but the Bogomol'nyi bound or the BPS bound. Its presence has been seen in the previous chapter where we found monopoles saturating this bound. In the first equation we see that if the central charges satisfy this upper bound some creation operators will fall out of the representations and it will become smaller. In the general case with n_0 central charges satisfying this upper bound the representations will shrink to $2^{2(n-n_0)}$ and if all $n/2$ central charges satisfy this bound we will again have a 2^n dimensional representation as in the massless case. In certain theories the fact that we know the number of states to be that of the short multiplet supersymmetry forces the central charges to satisfy the bound in order to keep the degrees of freedom fixed. It is instructive to work through the 4 dimensional case in order to identify the origin of these central charges. Starting with the center of mass frame $P_\mu = (M, 0, 0, 0)$, the N-extended supersymmetry algebra with central charges reads

$$\begin{aligned} \{Q_\alpha^I, \overline{Q}_{\dot{\beta} J}\} &= 2M\delta_J^I \delta_{\alpha\dot{\beta}} && (4.26) \\ \{Q_\alpha^I, Q_\beta^J\} &= \epsilon_{\alpha\beta} Z^{IJ} && (4.27) \end{aligned}$$

By restricting to $N = 2$ the central charges can be rewritten in terms of a single complex central charge. As was shown by D. Olive and E. Witten in 1978, this central charge can be written in terms of the electric and magnetic charge when present

$$Z^{IJ} = 2\epsilon^{IJ} Z = 2\phi_0 \epsilon^{IJ} (q_e + iq_m) \tag{4.28}$$

where ϕ_0 is the vacuum expectation value of some scalar field. The form is evidently not so surprising following the solitonic solutions of the previous chapter. Due to the presence of the central charges one favorably introduce two different sets of creation and annihilation operators fulfilling

$$a_\alpha = \frac{1}{\sqrt{2}}\left(Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right) \tag{4.29}$$

$$b_\alpha = \frac{1}{\sqrt{2}}\left(Q^1_\alpha - \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right) \tag{4.30}$$

With these operators the supersymmetry algebra takes the form

$$\{a_\alpha, a^\dagger_\beta\} = 2(M+ \mid Z \mid)\delta_{\alpha\beta} \tag{4.31}$$

$$\{b_\alpha, b^\dagger_\beta\} = 2(M- \mid Z \mid)\delta_{\alpha\beta} \tag{4.32}$$

The positivity condition now requires the right hand sides of this algebra to be non-negative. This is the Bogomol'nyi bound given by

$$M \geq \mid Z \mid \tag{4.33}$$

and can depending on taste be seen as an upper bound for the central charges or a lower bound for the mass. When the mass of a state fulfills this bound it is referred to as a BPS state. In that case the mass is completely given in terms of the charges of the system which in this 4 dimensional $N = 2$ case would read

$$M = \mid Z \mid = \phi_0\sqrt{q_e^2 + q_m^2} \tag{4.34}$$

In the case of a BPS state the half the algebra vanishes and we have again a short multiplet as was the case for the massless representations. We say that half the supersymmetry is broken but one should rather say that a BPS solution to a supersymmetric field theory preserves half the supersymmetry. The b_α's operators generate the unbroken supersymmetries while the broken supersymmetry operators, the a_α's, instead generate the supermultiplet. In higher dimensions we will see that the central charges are represented by anti-symmetric tensor fields and that BPS states are p-brane solutions which are charged under these anti-symmetric tensor fields.

4.3 Super–Yang–Mills

Yang–Mills theory can today describe a lot of observed phenomena in particle physics. The standard model, which is the best non-gravitational model in describing physical phenomena, is built upon a scalar field (Higgs) coupled to a Yang–Mills field, a so called Yang-Mills-Higgs system. The quantization procedure of Yang–Mills theory is built on perturbation theory where one expands in powers of the coupling constant. These powers are the same as the number of loops in the corresponding Feynman diagram. Here problems occur when studying phenomena of the strong interaction described by an $SU(3)$ gauge theory because the large coupling constant makes perturbation theory fail. For these strongly coupled theories the hope relies on duality. If the theory possesses a strong-weak duality which inverts the coupling constant, i.e. $g \to \frac{1}{g}$, it would imply that the strongly coupled theory could be retained by perturbation theory of the weakly coupled dual theory. In the Georgi–Glashow model we saw that there were indications that this theory indeed possessed these kind of duality properties but it was later shown that it did not. The basic feature opposing this duality was the fact that magnetic monopoles had spin 0 while the W bosons had spin 1 and could thus not be related through a duality transform. By extending this theory to a supersymmetric version this obstruction can be removed. The requirement are that the four-dimensional theory is fully extended to $N = 4$ supersymmetry. A brief attention will be put at the structure of these supersymmetric extended Yang–Mills theories.

4.3.1 $N = 1$, $D = 4$

To obtain the $N = 1$ SYM theory in 4 dimensions we begin by looking at the field content in the representation table 4.6 and find

$$\mathbf{1}_{-1} \oplus \mathbf{1}_{-1/2} \oplus \mathbf{1}_{1/2} \oplus \mathbf{1}_{1} = \lambda_\alpha \oplus A_a \tag{4.35}$$

where λ is a chiral spinor and A is the gauge potential. These can be combined into a vector superfield in the following form

$$V = \ldots - \theta\sigma^a\bar{\theta}A_a + i\theta^2(\bar{\theta}\bar{\lambda}) - i\bar{\theta}^2(\theta\lambda) + \frac{1}{2}\theta^2\bar{\theta}^2 D \tag{4.36}$$

where D is a auxiliary field. The gauge potential A_a is a non-abelian gauge field taken in the adjoint representation and thus all fields must lie in the adjoint representation of this gauge group. This vector superfield is often referred to as the prepotential because it is unconstrained and the true gauge potential can be derived from it. So by defining

$$W_\alpha = \frac{1}{8g}\bar{D}^2(e^{2gV}D_\alpha e^{-2gV}) \tag{4.37}$$

where g is the gauge coupling constant, one finds that

$$W = (-i\lambda + \theta D - i\sigma^{ab}\theta F_{ab} + \theta^2\sigma^a\nabla_a\bar{\lambda})(y) \tag{4.38}$$

where $y^a = x^a + i\theta\sigma^a\bar{\theta}$. From this the $N = 1$ SYM lagrangian can finally be obtained

$$-\frac{1}{4}\int d^4x d^2\theta \mathrm{tr} W^2 = \int d^4x \mathrm{tr}\left[-\frac{1}{4}F_{ab}F^{ab} + \frac{i}{4}F_{ab}\tilde{F}^{ab} - i\lambda\sigma^a\nabla_a\bar{\lambda} + \frac{1}{2}D^2\right] \tag{4.39}$$

The second term is of course a topological invariant giving, under integration, the instanton number. So by introducing the complex coupling constant due to Witten

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \tag{4.40}$$

the action can be rewritten the standard form for studying duality properties.

$$\frac{1}{16\pi}\mathrm{Im}\left[\tau\int d^4x d^2\theta \mathrm{tr} W^2\right] = \frac{1}{g^2}\int d^4x \mathrm{tr}\left[-\frac{1}{4}F_{ab}F^{ab} + \frac{g^2\theta}{32\pi^2}F_{ab}\tilde{F}^{ab} - i\lambda\sigma^a\nabla_a\bar{\lambda} + \frac{1}{2}D^2\right] \tag{4.41}$$

It can further be coupled to matter fields but this coupling is left out here.

4.3.2 $N = 2,\ D = 4$

Deriving lagrangians with higher order supersymmetry could be a tedious work. If one just starts with the representation and introduce all fields there is no general scheme in how one incorporates the fields in a supersymmetric manner. But a very nice and fruitful trick can be done, namely that of dimensional reduction. In short what one does is to look at an $N = 1$ theory in a higher dimension. These are all well known because the $N = 1$ theory looks pretty much the same in all dimensions. The dimension chosen is of course that which preserves the total number of supersymmetry generators, i.e. if we want $N = 2$ in four dimensions with a total of $2\times 4 = 8$ supersymmetry generators we need a space-time dimension with spinors of real spinor dimension 8. By looking at the dimensions in table 4.1 we find that both 5 dimensions and 6 dimensions have this spinor dimension, so we conclude that $N = 2,\ D = 4$ could be obtained from $N = 1,\ D = 6$ or $D = 5$.

Starting with the $N = 2$ representation we find

$$\mathbf{1}_{-1}\oplus\mathbf{2}_{-1/2}\oplus\mathbf{1}_0\oplus\mathbf{1}_0\oplus\mathbf{2}_{1/2}\oplus\mathbf{1}_1 = A_a\oplus\lambda_{\dot{\alpha}}\oplus\psi_\alpha\oplus\phi \tag{4.42}$$

So the field content is one gauge field A_a, two spinors $\lambda,\ \psi$ of different chirality and one complex scalar ϕ. Now this is exactly the field content one gets if one combines the $N = 1$ gauge field representation with the $N = 1$ matter representation, so one suspects that these could be put together in some way which would extend the supersymmetry to $N = 2$. This is in fact so and the way they should be put together is in the form

$$I = \mathrm{Im}\ \mathrm{tr}\int d^4x\frac{\tau}{16\pi}\left[\int d^2\theta W^2 + \int d^2\theta d^2\bar{\theta}\Phi^\dagger e^{-2gV}\Phi\right] \tag{4.43}$$

Of course the second supersymmetry is not manifest in this form and thus one would like to put this lagrangian in a more manifest $N = 2$ form. This can be done by introducing an $N = 2$ superfield built from a second set of spinor coordinates denoted $\tilde{\theta}$. The field looks like

$$\Psi = (\Phi + \sqrt{2}\tilde{\theta}W + \tilde{\theta}^2 G)(\tilde{y},\theta) \tag{4.44}$$

where $\tilde{y}^a = x^a + i\theta\sigma^a\bar{\theta} + i\tilde{\theta}\sigma^a\bar{\tilde{\theta}}$ and

$$G(\tilde{y},\theta) = -\frac{1}{2}\int d^2\bar{\theta}\Phi^\dagger e^{-2gV} \tag{4.45}$$

where $\Phi = \Phi(\tilde{y} - i\theta\sigma^a\bar{\theta},\theta,\bar{\theta})$ and $V = V(\tilde{y} - i\theta\sigma^a\bar{\theta},\theta,\bar{\theta})$. The lagrangian can now be put in the form

$$I = \mathrm{Im}\left[\frac{\tau}{16\pi}\int d^4x d^2\theta d^2\tilde{\theta}\frac{1}{2}\mathrm{tr}\Psi^2\right] \tag{4.46}$$

Performing the $d^2\tilde{\theta}$ integration gives us back the action in the form (4.43). Now by defining

$$\mathcal{F}(\Psi) := \frac{1}{2}\mathrm{tr}\tau\Psi^2 \tag{4.47}$$

the action can be rewritten into the form

$$I = \frac{1}{16\pi}\mathrm{Im}\int d^4x d^2\theta d^2\tilde{\theta}\mathcal{F}(\Psi) \tag{4.48}$$

Noticeable is that $\mathcal{F}$ only depends on Ψ and not on $\Psi^\dagger$. This is referred to as the holomorphicity condition of the prepotential, $\mathcal{F}$. So the requirement of $N = 2$ supersymmetry is transformed into the holomorphicity constraint of the prepotential. Performing the $d^2\tilde{\theta}$ integration now leaves us with

$$I = \frac{1}{16\pi}\mathrm{Im}\int d^4x\left[\int d^2\theta(W^2)^{ij}\mathcal{F}_{ij}(\Phi) + \int d^2\theta d^2\bar{\theta}(\Phi^\dagger e^{-2gV})^i\mathcal{F}_i(\Phi)\right] \tag{4.49}$$

where $\mathcal{F}_i(\Phi) := \frac{\partial\mathcal{F}(\Phi)}{\partial\Phi^i}$, $\mathcal{F}_{ij}(\Phi) := \frac{\partial^2\mathcal{F}(\Phi)}{\partial\Phi^i\partial\Phi^j}$ and i, j are gauge indices. Although this theory does not consist of magnetic monopoles with spin one, and thus can not possess a Montonen–Olive duality, Seiberg and Witten [30] have shown that it possess another striking duality property. The action can be shown to be form invariant under the duality transformation

$$\Phi_D = \mathcal{F}'(\Phi), \quad \mathcal{F}_D'(\Phi_D) = -\Phi \tag{4.50}$$

which can be seen as a Legendre transformation of the form $\mathcal{F}_D(\Phi_D) = \mathcal{F}(\Phi) - \Phi\Phi_D$ and then to be in fact $SL(2,\mathbb{Z})$ self dual, see [31]. Choosing a Higgs vacuum, say a with dual a_D the metric of the moduli space is given by

$$ds^2 = \mathrm{Im}(da_D d\bar{a}) \tag{4.51}$$

The symmetry breaking due to the specific choice of vacuum forces the solutions to be BPS states as they were contained in a short multiplet before the breaking and must remain so. The central charges of these states can be written

$$Z = an_e + a_D n_m \tag{4.52}$$

with mass formula

$$m^2 = |Z|^2 \tag{4.53}$$

Now the $SL(2,\mathbb{Z})$ transformations of the a's leaves this mass formula invariant. Seiberg and Witten went further to show that the it was preserved under quantum theory. They were also able to prove confinement in this $N = 2$ super Yang-Mills theory.

In paper I we looked at the dyon spectrum for $N = 2$ super Yang-Mills with higher gauge groups, especially $SU(3)$, coupled to matter multiplets. There we saw that this duality procedure is not in fact easily transferred to the general case and there are thus a lack evidence why this theory in fact should be $SL(2,\mathbb{Z})$ self-dual. In this paper we also derived the moduli space for the $(1,1)$ monopole configuration of the gauge group $SU(3)$. We found it to be

$$\mathbb{R}^3 \times \frac{S^1 \times Taub - NUT}{\mathbb{Z}_2} \tag{4.54}$$

4.3.3 $N = 4$, $D = 4$

The Lagrangian of $N = 4$ super Yang-Mills contains, in addition to the vector superfield V as before, also three chiral superfields Φ_i transforming according to the adjoint representation of the gauge group. It is given by

$$\mathcal{L} = \frac{1}{e^2}\int d^2\theta d^2\bar{\theta}\sum_{i=1}^{3}\bar{\Phi}_i e^{2V}\Phi_i + \frac{1}{8\pi}Im\left[\int d^2\theta\,\tau W^\alpha W_\alpha\right]$$

$$-\left[\int d^2\theta\sqrt{2}\Phi_1\Phi_2\Phi_3 + h.c.\right] \tag{4.55}$$

This theory can be proven to possess all the features of the proposed Montonen–Olive $SL(2,\mathbb{Z})$ duality. See [32] for details.

4.4 Supergravity

Next example of supersymmetric theories are those including gravity. These are called supergravity theories and are characterized by the existence of the metric tensor in the multiplet. After the development of higher-dimensional supersymmetry representations it was discovered that the four-dimensional superalgebra with higher N could be deduced from higher-dimensional superalgebras through dimensional reduction. Through the Kaluza-Klein ansatz there was a great effort put in trying to compactify higher-dimensional supergravity theories to four dimensions in hope of getting a realistic theory with low energy behavior as that of the standard model. It was early seen [33] that there was only one candidate in order for this to happen, namely $D = 11$ supergravity. This is so due to the fact that $D = 11$ is the upper limit for a theory consisting of no higher spin than 2 when compactified down to four dimensions and the lower limit for a space to leave us with gauge group $SU(3) \times SU(2) \times U(1)$ upon compactification. The first half of the 80's were spent in classifying compactifications for all kind of topologies of the internal space. But for this to be a fundamental theory describing all interaction a series of problems arose. One problem was that the supergravity theory is not renormalizable and thus although possessing beautiful symmetries it is still not a consistent quantum theory of gravity. Another problems were that chiral spinors could not be obtained through the common compactification procedure. There was also a problem in obtaining the right number of families upon compactification. The mass spectrum obtained through compactification were built in towers with all particles lying in the Planck mass regime and it was thus believed that all observed elementary particles must lie in the massless spectrum and acquire mass through quantum corrections. Two of these problems were later solved by Witten as he showed that chiral spinors can be obtained by compactifying over certain orbifolds. The right number of families can also be obtained by compactifying the obtained 10 dimensional theory a Calabi–Yau manifold with euler number 6. This is due to the discovery that the number of families are followed by the role

$$N_f = \frac{\chi(\mathscr{M})}{2}. \tag{4.56}$$

Another problem yet to be resolved is the so called *vacuum degeneracy problem* [34, 35] which is referred to as the problem of picking out one specific Calabi–Yau manifold from an infinite number. Today supergravity theories have been widely enriched through superstring theory and in the sequel of this section focus will be firmly put on those supergravity theories of special interest in string theory.

4.4.1 $D = 11$ supergravity

As will be the case for all supergravity theories treated here the field content will be read off the representation table 4.4. For the $N = 1$ supersymmetry representation in eleven dimensions containing the graviton the total spectrum looks like

$$\mathbf{44} \oplus \mathbf{84} \oplus \mathbf{128} = g_{mn} \oplus A_{mnp} \oplus \psi_\mu^m \tag{4.57}$$

With this field content Cremmer, Julia and Scherk were able to write down a supersymmetric lagrangian [36] describing the dynamics of the fields. The bosonic part of this action reads

$$I_{11} = \frac{1}{\kappa^2} \int d^{11}x \sqrt{-g} (R - \frac{1}{48} F_{(4)}^2) - \frac{1}{12} \int F_{(4)} \wedge F_{(4)} \wedge A_{(3)} \tag{4.58}$$

where F is the field strength of the potential A defined by $F_{(4)} := dA_{(3)}$.

It should be emphasized that the equations of motions, derivable from the action above, can be retained by imposing a couple of constraints to the fields in the superspace formulation. Here the equations of motion are obtained by solving the integrability conditions which are just the Bianchi identities of the fields in question. See [37] for details. This principle will also be the ground for possibility of treating p-brane dynamics in various supergravity theories through superembeddings. Here the equations of motions are obtained by imposing certain constraints on the embedding matrix, called embedding conditions. See paper II for details.

The $D = 11$ $N = 1$ supergravity theory has been deeply dissected [36, 38, 37] over the first half of the 80's. The biggest effort have perhaps been put in deriving the resulting spectra obtained through compactification over different compact internal manifolds. The main study has been compactifications down to four dimensions. Here the most interesting cases are internal manifolds with isometry group $SU(3)\times SU(2)\times U(1)$ or rather manifolds with isometry group containing this gauge group as a subgroup. The reason is of course that the gauge group of the standard model is obtained this way. All these manifolds have been classified and denoted $M(m,n)$, $m,n \in \mathbb{Z}$, where the integer numbers m,n is the topological classes of a $U(1)$ bundle over $\mathbb{C}P^2 \times S^2$, see [39]. These manifolds all have the property of being Einstein manifolds and all but two have isometry group $SU(3) \times SU(2) \times U(1)$. The other two are $M(1,0) = S^5 \times S^3$ and $M(0,1) = \mathbb{C}P^2 \times S^3$ which have isometry groups $SU(4) \times SU(2)$ and $SU(3) \times SU(2) \times SU(2)$ respectively. All these manifolds have full holonomy group $\mathscr{H} = SO(7)$ except for $M(3,2)$ with holonomy group $\mathscr{H} = SU(3)$. This means that it does not break all the supersymmetry upon compactification but there is a $N = 2$ supersymmetry left in four dimensions. Other manifolds of interest have been S^7 and T^7 because they are both parallelizable. This means that there is a global non-Levi–Civita connection with vanishing curvature which implies that the holonomy group is just the identity. Through the Killing spinor equation with this typical connection no supersymmetry is seen to be broken upon compactification. When considering compactifications it is worth noting that the concept of dimensional reduction is the same as compactification over the torus, here T^7, and just throwing away the massive modes.

4.4.2 $D = 10$ type IIA supergravity

Looking at the table of massless representations 4.4 restricting to ten dimensions and $N = (1,1)$ the field content looks like

$$\mathbf{35}_v \oplus \mathbf{56}_v \oplus \mathbf{28}_v \oplus \mathbf{8}_v \oplus \mathbf{1} = g_{mn} \oplus C_{(3)} \oplus B_{mn} \oplus C_{(1)} \oplus \phi \tag{4.59}$$

$$\mathbf{8}_s \oplus \mathbf{8}_c \oplus \mathbf{56}_s \oplus \mathbf{56}_c = \lambda_\mu \oplus \lambda_{\dot\mu} \oplus \psi^m_\mu \oplus \psi^m_{\dot\mu} \tag{4.60}$$

These fields can be incorporated into a supersymmetric action which is called type IIA supergravity of which the bosonic part reads

$$
\begin{aligned}
I_{IIA} &= \frac{1}{\kappa^2}\int d^{10}x\sqrt{-g}\Big(e^{-2\phi}[R+4(\partial\phi)^2-\frac{1}{12}H_{(3)}^2]- \\
&\quad -[\frac{1}{2}R_{(2)}^2+\frac{1}{24}R_{(4)}^2]\Big)-\frac{1}{4}\int dC_{(3)}\wedge dC_{(3)}\wedge B_{(2)},
\end{aligned}
\tag{4.61}
$$

with $H_{(3)} = dB_{(2)}$, $R_{(2)} = dC_{(1)}$ and $R_{(4)} = dC_{(3)} + H_{(3)} \wedge C_{(1)}$.

The name type IIA supergravity originates from type IIA superstring theory [40]. The background in which type IIA superstring theory propagates is namely type IIA supergravity. To be more precise, type IIA supergravity is the low energy background of type IIA superstring theory to first order in α'.

4.4.3 $D = 10$ type IIB supergravity

Type IIB theory is chiral [41, 42, 43] so from the representation table 4.4 for $D = 10$ with $N = (2, 0)$ instead the field content reads

$$
\begin{aligned}
(\mathbf{35}_v)_0\oplus(\mathbf{28}_v)_2\oplus\mathbf{1}_4 &= g_{mn}\oplus B_{mn}\oplus\phi && (4.62)\\
\mathbf{1}_{-4}\oplus(\mathbf{28}_v)_{-2}\oplus(\mathbf{35}_-)_0 &= C_{(0)}\oplus C_{(2)}\oplus C_{(4)}^+ && (4.63)\\
(\mathbf{8}_s)_{-3}\oplus(\mathbf{56}_s)_{-1}\oplus(\mathbf{56}_s)_1\oplus(\mathbf{8}_s)_3 &= \lambda_\mu\oplus\tilde\lambda_\mu\oplus\psi_\mu^m\oplus\tilde\psi_\mu^m && (4.64)
\end{aligned}
$$

This theory contains a self-dual 4-form so there is no action formulation of the theory. But excluding this field for now, an action for the rest of the fields can be written down. This theory contain an $SL(2,\mathbb{R})$ self-duality so the action will be written in such a transparent way as possible in order to make this self-duality of the type IIB theory manifest. (For more discussions regarding various dualities in these different supergravity theories see chapter 6.) The action for the rest of the fields looks like

$$
I_{IIB} = \frac{1}{\kappa^2}\int d^{10}x\sqrt{-g}(R-\frac{1}{12}H_{mnp}^T M H^{mnp}+\frac{1}{4}tr(\partial^m M\partial_m M^{-1})), \tag{4.65}
$$

where

$$
H = d\tilde B = \begin{pmatrix} dB_{(2)} \\ dC_{(2)} \end{pmatrix}; \quad M = e^{\phi}\begin{pmatrix} |\tau|^2 & C_{(0)} \\ C_{(0)} & 1 \end{pmatrix}; \quad \tau = C_{(0)} + ie^{\phi}. \tag{4.66}
$$

This can be seen to be invariant under a global $SL(2,\mathbb{R})$ transformation. Introducing $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ the transformations

$$
M \to \Lambda M \Lambda^T \equiv \tau \to \frac{a\tau+b}{c\tau+d}; \quad \text{and} \quad \tilde B \to (\Lambda^T)^{-1}\tilde B \tag{4.67}
$$

can be seen to leave I_{IIB} invariant. This duality is of course preserved on inclusion of the self-dual 4-form and the fermion fields.

4.4.4 $D = 10$ type I supergravity

Following the same procedure and by looking at the representation table bearing in mind this case only possesses $N = 1$ supersymmetry one finds a field content of

$$\begin{aligned} \mathbf{35}_v \oplus \mathbf{28}_v \oplus \mathbf{1} &= g_{mn} \oplus B_{mn} \oplus \phi && (4.68) \\ \mathbf{8}_s \oplus \mathbf{56}_c &= \lambda_\mu \oplus \psi^m_{\dot\mu} && (4.69) \end{aligned}$$

From the representation table 4.4 it is obvious though that there is also a $N = 1$ Super–Yang–Mills representation. This is often used to derive $N = 4$ Super–Yang–Mills in $D = 4$ by dimensional reduction. What is striking here is that this SYM representation can be coupled to the supergravity multiplet. From the point of view of pure supergravity one could of course have left out this SYM representation but as there is no doubt that the basic interest are put at various string theories, the type I supergravity theory is the low energy effective background of that theory. The $N =$ SYM multiplet can be read off from the same representation table 4.4 and looks like

$$\mathbf{8}_v \oplus \mathbf{8}_c = A_m \oplus \lambda_{\dot\mu} \tag{4.70}$$

The coupling between these theories was worked out already in (82), see [44, 45, 46]. Before writing down the bosonic part of the action describing the dynamics of these fields it must be said that there actually are two $N = 1$ supergravity theories coupled to SYM. The first with above field representation is called heterotic supergravity and have a bosonic action which looks like

$$I_{het} = \frac{1}{\kappa^2}\int d^{10}x\sqrt{-g}e^{-\phi}\left(R + 4(\partial\phi)^2 - \frac{1}{12}H^2_{(3)} - \frac{1}{4}trF^2\right), \tag{4.71}$$

Here $H_{(3)} = dB_{(2)}$ and $F = dA + A\wedge A$, and A transforms in the adjoint representation of $SO(32)$ or $E_8 \times E_8$. The other supergravity multiplet with the field content but with different dilaton coupling is called the type I supergravity multiplet. Its bosonic action reads

$$I_I = \frac{1}{\kappa^2}\int d^{10}x\sqrt{-g}\left(e^{-2\phi}[R + 4(\partial\phi)^2] - \frac{1}{12}R^2_{(3)} - \frac{1}{4}e^{-\phi}trF^2\right), \tag{4.72}$$

Here $R_{(3)} = dC_{(2)}$ and $F = dA + A \wedge A$ is the field strength of an $SO(32)$ gauge field. The appearing of these typical gauge groups is due to anomaly cancellation in superstring theory.

5

String Theory

In ordinary quantum field theory, elementary particles are described through the Feynman diagrams as pointlike objects. These theories are therefore often plagued with ultraviolet divergences due to the extreme locality of the particles. In a theory where the fundamental objects no longer are pointlike but instead higher-dimensional, these ultraviolet divergences disappear because the extreme locality of a point particle vertex is exchanged to a tube-like vertex with no remaining unique interaction point. In string theory ordinary particles are reinterpreted as vibrational modes of the string. This makes string theory a very pleasant generalization of ordinary field theory. String theory from a purely mathematical point of view originated a long time ago with the study of minimal surfaces including strings, and was well classified already by Gauss. In physical theories the entry of particles built from higher-dimensional objects was due to Dirac in 1950, when he discussed the matter of infinite energies in the point particle models and proposed strings and membranes as the solution to this problem. He was the first to write down an action describing the dynamics of these objects. Later at the end of the 60's Veneziano was trying to describe the strong interaction with something called the dual resonance model which later was shown to be describing the scattering of open bosonic strings. This was noticed together with the real entry of string theory due to Ramond, Neveu and Schwarz when they introduced the fermionic string model in 1974, which independently was discovered by Gol'fand and Likhtman already in 1971. At this time string theory had no intentions of trying to describe the physical world in a unified manner but was merely a model of the strong interaction. In 1973 though, QCD was proposed and gave better predictions than the string model so string theory was widely abandoned. I was not until 1984 that string theory reappeared now as a possible theory for describing all forces in the universe. This period of time (1984-1985) are usually referred to as the *first string revolution*. Here an anomaly cancellation mechanism was found when coupled to the gauge groups

$SO(32)$ or $E_8 \times E_8$ and the two heterotic superstring theories were discovered. But maybe the most encouraging result was that upon compactification down to four dimensions one could, by compactifying over certain Calabi–Yau manifolds, obtain not only the gauge group of the standard model but also the right number of families. The Calabi–Yau manifolds preserves only one quarter of the supersymmetry so the obtained theory in four dimensions is a $N = 1$ supersymmetric theory with three generations of chiral fermions including gravity. The problem persisting was merely the *vacuum degeneracy problem* and was basically the fact that there are too many possible vacuum configurations or in other words Calabi–Yau manifolds. This is not even up to this date completely resolved. To summarize one was left with five consistent superstring theories which from a unified point of view looks a bit disturbing, but in comparison with the merely infinite number of theories available outside the superstring regime this was nevertheless quite satisfying. The massless spectra of these theories will be derived in the following sections.

The *second string revolution* came in the mid 90's when all these consistent string theories were put on equal footing through the eyes of duality. Here the former $D = 11$ supergravity theory was reinstated as a part of the total M-theory of which all these theories were merely some perturbative regime. These dualities will be discussed in next chapter where probes of the dualities in terms of p-branes are introduced. So string theory does in fact propose higher-dimensional objects in the spectrum as solutions to the background field equations. Interesting to note is that string theory contains a minimal length $\sim \sqrt{\alpha'}$ which can be found from the modified Heisenberg's uncertainty principle, namely

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' \frac{\Delta p}{\hbar} \tag{5.1}$$

The modification of Heisenberg's uncertainty principle is due to the fact that strings become larger at higher energies which is not the case for a point particles, see [47, 48]. This effect can most easily be seen through T-duality, a duality which will be presented in next chapter.

This chapter contains two sections of which the first is subjected to the bosonic string theory which, although not consistent due to the presence of tachyons in the spectrum, contains a lot of information which is also true in the superstring theory. In section two the five consistent superstring theories are exploited where for instance their massless spectra are derived. For a more detailed study see Refs. [49, 50, 51, 52, 53, 54, 55].

5.1 Bosonic String

String theory is basically a generalization of the notion of point particles. Here the fundamental particles are replaced by one-dimensional strings which through their propagation in time sweeps out what is called the two-dimensional world-sheet. The dynamics of the string is derived from the principle of minimizing the area of the world-sheet in a similar way as an ordinary point particle is described by minimizing the length of its world-line. The action describing this kind of motion is first due to Dirac-Nambu-Goto [56, 57] and reads

$$S_{DNG} = T \int_\Sigma d^2\sigma \sqrt{-det(\partial_m X^{\underline{m}} \partial_n X^{\underline{n}} \eta_{\underline{mn}})}. \tag{5.2}$$

This action functional simply measures the total volume of the world-sheet. Unfortunately the square root in the action makes it very hard to handle mathematically, especially upon quantization. It is therefore more suitable to rewrite the action in the following form

$$S_{BDH} = -\frac{T}{2} \int_\Sigma d^2\sigma \sqrt{h} h^{mn} \partial_m X^{\underline{m}} \partial_n X_{\underline{m}}. \tag{5.3}$$

which was originally done by Brink, Di Vechia and Howe [58]. Although the action is put on a mathematically more pleasant form, the drawback is of course the introduction of the auxiliary world-sheet metric. By introducing auxiliary fields into an action the theory receives additional algebraic constraints. On shell will the additional constraint in this case only lead to a coupling between the auxiliary world-sheet metric being and the induced metric of the embedding. When quantizing the theory, though, the fields are put arbitrarily off shell which leads to new problems. So although the action in this form is quadratic in the x's which makes the normal quantization procedure work, it is basically quantum mechanically inequivalent to the Dirac–Nambu–Goto theory. This a typical example of the arbitrariness in the quantization procedure which must be seen as a major drawback in today's understanding of quantum field theory. In this case there are two classically equivalent theories with different quantum theories. There is actually one target space dimension in which they are equal and this is the same dimension in which the quantum theory of the BDH action is conformal invariant, and that is $D = 26$.

Before digging any deeper into the features of the quantized bosonic string let us take a look at the classical dynamics of the string first. Recall that a metric in two dimension is always locally conformally flat which means $h_{mn} \equiv e^\Lambda \eta_{mn}$. The BDH action can additionally be seen to be conformally invariant, and in what is called a conformal gauge the action can be rewritten in the form

$$S_{cg} = -\frac{T}{2} \int_\Sigma d^2\sigma \eta^{mn} \partial_m X^{\underline{m}} \partial_n X_{\underline{m}}. \tag{5.4}$$

From this action the equations of motion of the bosonic string can most easily be derived. The string dynamics is seen to be described by the ordinary wave equation

$$\Box X^{\underline{m}} = 0, \tag{5.5}$$

whose solution contains two different parts, called left and right moving modes. The solution, X, can thus be decomposed into its left and right moving parts.

$$X^{\underline{m}}(\tau,\sigma) = X_L^{\underline{m}}(\sigma+\tau) + X_R^{\underline{m}}(\sigma-\tau). \tag{5.6}$$

As the wave equation is a differential equation of order two, two boundary conditions must be imposed to solve it exactly. There are three different types of boundary conditions namely

String type	*Type*	*Boundary*		*condition*
Closed	*Periodic*	$X^{\underline{m}}(\sigma+2\pi)$	$=$	$X^{\underline{m}}(\sigma)$
Open	*Dirichlet*	$X^{\underline{m}}(0)$	$=$	$X^{\underline{m}}(\pi)=0$
	Neumann	$\partial_\sigma X^{\underline{m}}(0)$	$=$	$\partial_\sigma X^{\underline{m}}(\pi)=0$

(5.7)

For the closed strings with periodic boundary conditions the solution can be written

$$X^{\underline{m}}(z,\bar{z}) = q^{\underline{m}} - \frac{i}{4}\alpha' p^{\underline{m}} ln(z\bar{z}) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^{\underline{m}} z^{-n} + \tilde{\alpha}_n^{\underline{m}}\bar{z}^{-n}\right) \tag{5.8}$$

where a Wick rotation has been done and complex coordinates defined by $z := e^{(\tau+i\sigma)}$ have been introduced. The open string solution with Neumann boundary condition only contains half the oscillator degrees of freedom due to an equivalence between left and right moving modes. Its solution have the different look

$$X^{\underline{m}}(z,\bar{z}) = q^{\underline{m}} - i\alpha' p^{\underline{m}} ln(z\bar{z}) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^{\underline{m}}}{n}\left(z^{-n} + \bar{z}^{-n}\right). \tag{5.9}$$

The situation with Dirichlet boundary conditions will be left to chapter 7 where the concept of D-branes is introduced.

Looking at the quantum theory of the bosonic string there are a couple of different approaches, all of which pick out the dimension of the target space to be 26 in order for the theory to be consistent. Some typical problems encountered that pick out this dimension are

- In light-cone gauge the closure of the Lorentz generators at the quantum level is only possible in 26 dimensions.
- In the old covariant quantization approach a ghost free spectrum can only be obtained for $D \leq 26$ and unitarity at higher loop levels restricts the dimension further, to 26.
- In the path integral formalism imposing conformal invariance at the quantum level implies $D \leq 26$ where $D = 26$ is a critical dimension which decouples the so called Liouville field (the conformal mode of the metric) from the action. In all other cases the Liouville field becomes a dynamical variable.

So the first problem with bosonic string theory at the quantum level is that it only is consistent in a target space with 26 dimensions. Another problem arises from the spectrum of the theory, the lowest state is namely a tachyon, i.e., a state with negative mass square. Tachyon states were discussed in previous chapter, where they were referred to as unphysical (they travel faster than the speed of light) and can thus not be accepted in a physical theory. In that sense bosonic string theory is non-physical and must be discarded. In previous chapter though the introduction of supersymmetry guaranteed that these negative norm states vanished from the spectrum. So this problem can be sorted out if there exists a supersymmetric generalization of the bosonic string theory. This theory does indeed exist and goes under the name of superstring theory. Next section treats superstring theory but, although unphysical, bosonic string theory gives a lot of important insights even to superstring theory so there are a couple features that ought to described before turning the attention to superstrings. The massless spectrum of bosonic string theory, which is the first excited state because the vaccum is a tachyon, will differ for closed strings and open strings. For the open bosonic string with only one mode the massless sector will contain only a vector field $A_{\underline{m}}$. The closed bosonic string with two separate modes have a broader massless sector with a full 2-tensor in the spectrum. This tensor can be decomposed into irreducible parts which are the traceless symmetric part, $g_{\underline{mn}}$, the anti-symmetric part, $B_{\underline{mn}}$ and the trace part, ϕ. These are referred to as the metric, the two-form and the dilaton respectively. The content of the massless spectrum makes it possible to couple the closed strings to these fields. The closed string can thus be coupled to the metric, the two form and the dilaton field [59]. The couplings of the first two fields are easy and look like

$$S_g = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{h}h^{mn}\partial_m X^{\underline{m}}\partial_n X^{\underline{n}} g_{\underline{mn}}, \tag{5.10}$$

$$S_B = -\frac{1}{4\pi\alpha'}\int d^2\sigma\epsilon^{mn}\partial_m X^{\underline{m}}\partial_n X^{\underline{n}} B_{\underline{mn}}. \tag{5.11}$$

The dilaton coupling is not as transparent but from [59] one obtains

$$S_\phi = \frac{1}{4\pi}\int d^2\sigma\sqrt{h}R^{(2)}\phi, \tag{5.12}$$

where $R^{(2)}$ is the two-dimensional Ricci scalar. This last term is itself not conformal invariant, as is the case for the other terms, but as it is of higher order in α' it enables for a conformally invariant quantum theory. In the two-dimensional theory these background fields, $g_{\underline{mn}}$, $B_{\underline{mn}}$ and ϕ, are considered as coupling constants and conformal invariance at the quantum level forces β-functions of these coupling constants to vanish. For details see [59]. As these coupling constants merely are fields these β-functions are referred to as β-functionals. These can be calculated and are to lowest order in α'

$$\beta^{(g)}_{\underline{mn}} = R_{\underline{mn}} - \frac{1}{4}H_{\underline{mpq}}H_{\underline{n}}{}^{\underline{pq}} + 2\nabla_{\underline{m}}\nabla_{\underline{n}}\phi + \mathcal{O}(\alpha'), \tag{5.13}$$

$$\beta^{(B)}_{\underline{mn}} = \frac{1}{2}\nabla^{\underline{p}}H_{\underline{pmn}} - (\nabla^{\underline{p}}\phi)H_{\underline{pmn}} + \mathcal{O}(\alpha'), \tag{5.14}$$

$$\beta^{(\phi)} = -R + \frac{1}{12}H^2 - 4\nabla^{\underline{p}}\nabla_{\underline{p}}\phi + 4\nabla^{\underline{p}}\phi\nabla_{\underline{p}}\phi + \mathcal{O}(\alpha'), \tag{5.15}$$

where $R_{\mu\nu}$ is the Ricci tensor of space-time and $H_{(3)}$ is the field strength of the two form, i.e. $H_{(3)} = dB_{(2)}$. So as conformal invariance at the quantum level forced these β-functionals to vanish it is seen that the quantized string makes the background fields dynamical. The dynamics is to lowest order in α' described by the Einstein equations of motion. These equations of motion can equivalently be obtained by minimizing the action

$$I = \frac{1}{2\kappa^2}\int d^{26}X\sqrt{g}e^{-2\phi}(R + 4(\partial\phi)^2 - \frac{1}{12}H^2 + \mathcal{O}(\alpha')), \tag{5.16}$$

where κ is the gravitational coupling constant. Keeping only the first order terms this action is referred to as the low energy effective action for the background fields, in which the closed bosonic string can propagate.

5.2 Superstring

Extending the bosonic string to a supersymmetric theory will undoubtedly bring up questions regarding the embedding procedure associated with an area minimizing action. From the study of p-branes which is a generalization of the string concept, it is clear that supersymmetry implies supersymmetry. By this is meant that if we start with a superstring which possesses world-sheet supersymmetry, it can only propagate in a supersymmetric background. Now there are three different approaches to handle the embedding procedure of the string.

- The Neveu–Schwarz–Ramond (NSR) approach [60, 61, 62, 63], where one starts with manifest world-sheet supersymmetry and embeds it into ordinary target space. The target space supersymmetry arises through the so called GSO projection which is required for modular invariance and for obtaining the right fermionic degrees of freedom.

- The Green–Schwarz (GS) approach [64, 65], where instead the metric from a manifest supersymmetric target space is pulled back to an ordinary world-sheet. The world-sheet supersymmetry is obtained through a symmetry known as κ-symmetry which reduces the number of supersymmetry generators of target space by half.

- The doubly supersymmetric approach, where both the world-sheet and the target space is taken to be manifestly supersymmetric (superspace formulated).

> The embedding procedure is done through a supersymmetric extension known as superembedding. The right degrees of freedom is here obtained through the so called embedding condition which paper II for instance deals with quite thoroughly. For more references on the subject see paper II.

This procedure restricts the possible target space dimensions already at the classical level. The superstring can at the classical level only exist for $D = 3, 4, 6, 10$. Following the procedure of the bosonic string the conformal anomalies can only be removed for $D = 10$. We say that the critical dimension of the superstring is 10. See [66].

There are five different consistent superstring theories which will be looked at. By consistent is meant quantum mechanically consistent, i.e., they do not have any anomalies. The classical action for the superstring in the NSR approach in Wick rotated form with complex coordinates reads

$$S = -\frac{1}{4\pi\alpha'}\int d^2z(\bar{\partial}X^{\underline{m}}\partial X_{\underline{m}} - \psi^{\underline{m}}\bar{\partial}\psi_{\underline{m}} - \bar{\psi}^{\underline{m}}\partial\bar{\psi}_{\underline{m}}) \tag{5.17}$$

where $\psi^{\underline{m}}$ and $\bar{\psi}^{\underline{m}}$ are the components of a two dimensional Majorana spinor in two dimensions, i.e.

$$\psi^{\underline{m}} = \begin{pmatrix} \psi^{\underline{m}} \\ \bar{\psi}^{\underline{m}} \end{pmatrix} \tag{5.18}$$

The equations of motions for the fermionic fields become

$$\bar{\partial}\psi^{\underline{m}} = 0 \tag{5.19}$$

$$\partial\bar{\psi}^{\underline{m}} = 0 \tag{5.20}$$

which implies that their components are holomorphic and anti-holomorphic functions respectively. To solve these equations of motion two boundary conditions are needed. Again there are some differences between the open and closed sectors, so they are treated separately. By varying the action the total derivatives of the fermionic fields leads to a boundary term in the form $\psi^{\underline{m}}\delta\psi_{\underline{m}} - \bar{\psi}^{\underline{m}}\delta\bar{\psi}_{\underline{m}}$. The boundary conditions must be imposed such that this boundary term vanishes. For the open string there are two possiblities, namely

$$\begin{array}{lllll} \psi^{\underline{m}}(0) = \bar{\psi}^{\underline{m}}(0) & \psi^{\underline{m}}(\pi) = \bar{\psi}^{\underline{m}}(\pi) & (\mathrm{R}) \\ \psi^{\underline{m}}(0) = \bar{\psi}^{\underline{m}}(0) & \psi^{\underline{m}}(\pi) = -\bar{\psi}^{\underline{m}}(\pi) & (\mathrm{N}S) \end{array} \tag{5.21}$$

The first type of boundary condition is called Ramond (R) boundary condition and the second Neveu-Schwarz (NS) boundary condition. The closed string on the other hand has two independent sectors in the left and right moving modes. Here one can therefore have periodic (R) and anti-periodic (NS) boundary conditions in the left and right moving sectors independently. This leaves us with a total of four different sectors namely: R-R, R-NS, NS-R and NS-NS. The massless spectrum of the superstring will differ between the various cases but all will be built from the same constituents. The tachyon in the spectrum is removed by demanding supersymmetry in target space. There were three different approaches to obtaining this of which the above is the (NSR) approach with manifest world-sheet supersymmetry. Here one must impose the so called GSO projection in order to remove the tachyon and obtain a supersymmetric background. The other approaches will equivalently lead to the same supersymmetric background. So the ground state is in fact a target space spinor (or cospinor) lying in the Ramond sector, and there is a vector lying in the NS sector, i.e.

$$\mathbf{8}_v \oplus \mathbf{8}_s, \quad \text{or} \quad \mathbf{8}_v \oplus \mathbf{8}_c \tag{5.22}$$

for the open string. For the closed string the left and right moving modes are treated separately which brings the possible representations

$$(\mathbf{8}_v \oplus \mathbf{8}_s) \otimes (\mathbf{8}_v \oplus \mathbf{8}_s) \quad \text{or} \quad (\mathbf{8}_v \oplus \mathbf{8}_s) \otimes (\mathbf{8}_v \oplus \mathbf{8}_c) \tag{5.23}$$

of its spectrum. The first representation above is that of $N = 1$ SYM in $D = 10$ and will not give a consistent string theory by its own. The two latter cases leads both to consistent string theories with $N = 2$ target space supersymmetry although the first is chiral, $N = (2, 0)$, and the latter is non-chiral, $N = (1, 1)$. These will be referred to as type IIB and type IIA string theory respectively.

5.2.1 Type II superstrings

As was previously seen the representations of type IIA and type IIB superstrings were those of

$$\begin{array}{lcll} (\mathbf{8}_v \oplus \mathbf{8}_s) & \otimes & (\mathbf{8}_v \oplus \mathbf{8}_c), & \text{(IIA)} \\ (\mathbf{8}_v \oplus \mathbf{8}_s) & \otimes & (\mathbf{8}_v \oplus \mathbf{8}_s). & \text{(IIB)} \end{array} \tag{5.24}$$

To investigate the field contents of these representations they are decomposed into their irreducible parts. Both type IIA and type IIB have a common NS-NS sector [64] which can be decomposed into

$$\mathbf{8}_v \otimes \mathbf{8}_v = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v = \phi \oplus B_{(2)} \oplus g_{\mu\nu} \tag{5.25}$$

This spectrum is identical to that of the bosonic string, but now in a different target space dimension. It thus contains one scalar field referred to as the dilaton, one abelian two-form and a metric. These are of course all bosonic but are not the only bosonic constituents of the spectrum. There are also additional bosonic degrees of freedom coming from the R-R sector and are different for type IIA and type IIB.

$$\begin{array}{lclcll} \mathbf{8}_c \otimes \mathbf{8}_s & = & \mathbf{8}_v \oplus \mathbf{56}_v & = & C_{(1)} \oplus C_{(3)}, & \text{(IIA)} \\ \mathbf{8}_s \otimes \mathbf{8}_s & = & \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_s & = & C_{(0)} \oplus C_{(2)} \oplus C^{+}_{(4)}, & \text{(IIB)} \end{array} \tag{5.26}$$

So there is a one-form and a three-form in the type IIA spectrum and a scalar, a two-form and a self-dual four-form in the type IIB spectrum. (One should rather say a four-form with a self dual field strength $*dC^+_{(4)} = dC^+_{(4)}$.) The fermionic degrees of freedom are obtained from the NS$\otimes$R and R$\otimes$NS sectors.

$$\begin{array}{rcccl} \mathbf{8}_v \otimes \mathbf{8}_c & = & \mathbf{8}_s \oplus \mathbf{56}_c & = & \lambda_\alpha \oplus \psi^\mu_{\dot{\alpha}}, \\ \mathbf{8}_v \otimes \mathbf{8}_s & = & \mathbf{8}_c \oplus \mathbf{56}_s & = & \lambda_{\dot{\alpha}} \oplus \psi^\mu_{\alpha}. \end{array} \tag{5.27}$$

Put together these representations are seen to be nothing but those of $N = (1,1)$ and $N = (2,0)$, $D = 10$ supergravity (also called type IIA and type IIB supergravity). The dynamics of these field are again obtained by requiring conformal invariance of the string action at the quantum level. This puts constraints on the β-functionals which must vanish. As was seen in the bosonic string theory this implication lead to Einstein equations of motion and so is the case for superstring theory as well. To lowest order in α' these equations of motion are derivable from the supergravity actions presented in previous chapter. The type IIA and type IIB supergravity actions are therefore referred to as the low energy effective actions of type IIA and type IIB superstring theory.

5.2.2 Type I superstring

From the supergravity representations in $D = 10$ there was also seen to exist an $N = 1$ supergravity representation in $D = 10$. A not too wild guess would therefore make us believe that there should exist a type I superstring with $N = 1$ target space supersymmetry. Although not that easily obtainable there indeed exists a type I superstring [46] which can be derived from the type IIB superstring by projecting out half of the degrees of freedom in such a way that only the left-right symmetric parts remain. The projection operator contains an orientifold operation often denoted Ω which reverses the role of the left and right moving sectors. This will lead to an unoriented theory and its spectrum can be obtained from that of type IIB by keeping left-right symmetric parts. These are the representations of the graded symmetric parts, i.e.

$$(\mathbf{8}_v \otimes \mathbf{8}_s)\hat{\odot}(\mathbf{8}_v \otimes \mathbf{8}_s) = (\mathbf{8}_v \odot \mathbf{8}_v) \oplus (\mathbf{8}_s \wedge \mathbf{8}_s) \oplus (\mathbf{8}_v \otimes \mathbf{8}_s), \tag{5.28}$$

The field content of these representations is

$$\begin{array}{rcccl} (\mathbf{8}_v \odot \mathbf{8}_v) & = & \mathbf{1} \oplus \mathbf{35}_v & = & \phi \oplus g_{\mu\nu}, \\ (\mathbf{8}_s \wedge \mathbf{8}_s) & = & \mathbf{28} & = & C_{(2)}, \\ (\mathbf{8}_v \otimes \mathbf{8}_s) & = & \mathbf{8}_c \oplus \mathbf{56}_s & = & \lambda_{\dot{\alpha}} \oplus \psi^\mu_{\alpha}. \end{array} \tag{5.29}$$

and again these are seen to be exactly those of the type I supergravity representation seen in table 4.4. This theory is not consistent by itself though because it contains conformal anomalies. These anomalies can be getting rid off by including an open string sector, which lies in the representation of $N = 1$ SYM in $D = 10$, with Chan-Paton gauge group $SO(32)$. This means that the open string carries an $SO(32)$

charge at its endpoints. Put together the theory is consistent and the dynamics of these background fields can, in the low energy limit, be obtained from the type I supergravity action presented in previous chapter. See [66] for more details.

5.2.3 Heterotic superstrings

There are yet two other consistent superstring theories, namely the two heterotic superstrings with gauge groups $SO(32)$ and $E_8 \times E_8$ respectively, see Refs. [67, 68]. The basic features of the heterotic strings come from the fact that the left and right moving modes are independent of each other (except for the zero modes) and can thus be treated separately. In fact the right moving modes can be let to be those of a 10-dimensional superstring while instead the left moving modes are let to be those of a bosonic string in 26 dimensions compactified in a certain way down to 10 dimensions. The resulting bosonic modes from such a compactification will split into 10 transverse and 16 parallel modes of the compactification manifold. Here the 10 transverse bosonic coordinates will end up in a $\mathbf{8}_v$ representation and the other 16 coordinates will parametrize the internal compactification manifold. The key issue is therefore to classify all kinds of internal manifolds which are consistent with the imposing of conformal invariance at the quantum level. In fact there is only one type of internal manifolds possible and that is the tori constructed through

$$T = \mathbb{R}^{16}/\Lambda_{16} \tag{5.30}$$

where Λ must be an even Euclidean self-dual lattice. The requirement of a Euclidean self-dual lattice is the same as the requirement of modular invariance. Only two such 16-dimensional lattices exist and those are the weight lattice of $Spin(32)/\mathbb{Z}_2$ and the root lattice of $E_8 \times E_8$. These possible cases lead to the only two consistent heterotic superstrings and the representations of these groups are taken to be the adjoint representation and the $(\mathbf{248}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{248})$ respectively, both having the dimension 496. Combining this with the left-moving modes gives us the massless spectrum of the two possible heterotic superstrings, namely

$$(\mathbf{8}_v \oplus \mathbf{8}_c) \quad \otimes \quad (\mathbf{8}_v \oplus \mathbf{adj}_{SO(32)}) \tag{5.31}$$
$$(\mathbf{8}_v \oplus \mathbf{8}_c) \quad \otimes \quad (\mathbf{8}_v \oplus [(\mathbf{248}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{248})]) \tag{5.32}$$

where

$$(\mathbf{8}_v \oplus \mathbf{8}_c) \otimes \mathbf{8}_v = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v \oplus \mathbf{8}_s \oplus \mathbf{56}_c = \phi \oplus B_{(2)} \oplus g_{\mu\nu} \oplus \lambda_\alpha \oplus \psi^\mu_{\dot\alpha} \tag{5.33}$$

is the representation of N=1 supergravity in D=10 seen in Table 4.4 and

$$(\mathbf{8}_v \oplus \mathbf{8}_c) \quad \otimes \quad \mathbf{adj}_{SO(32)} \tag{5.34}$$
$$(\mathbf{8}_v \oplus \mathbf{8}_c) \quad \otimes \quad [(\mathbf{248}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{248})] \tag{5.35}$$

are the representations of N=1 super Yang-Mills in D=10 with gauge groups $SO(32)$ and $E_8 \times E_8$ respectively. These are also found in Table 4.4. Noticeable is that the

Theory	Type IIA	Type IIB	Type I	Het $SO(32)$	Het $E_8 \times E_8$
World-sheet topology	closed oriented	closed oriented	open, closed unoriented	closed oriented	closed oriented
Gauge group type	-	-	$SO(32)$ Chan-Paton	$SO(32)$ Cur. alg.	$E_8 \times E_8$ Cur. alg.
Susy	$N = (1,1)$	$N = (2,0)$	$N = 1$	$N = 1$	$N = 1$
Susy gen.	32	32	16	16	16
Irred. reps size	$2^8 = 256$	$2^8 = 256$	$8 \times 2^4 = 128$ $2^4 \times 496$	$8 \times 2^4 = 128$ $2^4 \times 496$	$8 \times 2^4 = 128$ $2^4 \times 496$

Table 5.1: The five consistent perturbative superstring theories

spectrum of the $SO(32)$ heterotic string is precisely that of the type I superstring although the gauge group here does not arise from charged endpoints as there are no open strings in the heterotic theory. Nevertheless these two theories will be seen to be dual to each other in the upcoming chapter.

5.2.4 Superstring summary

So we have seen that there are five consistent superstring theories. A sum up is made in Table 5.1 which contains the topology and the irreducible representation sizes etc. Although the above treatment was done purely through the NSR approach it must be stressed that the doubly supersymmetric approach is the most natural in a geometrical picture. This approach is the only one which trivially generalizes to objects of higher dimension, p-branes. The GS approach will also work but the κ-symmetry will be more difficult to see than in the doubly supersymmetric approach. The branescan which will be discussed in next chapter will be an extension of the possible target space dimensions for various branes. For the superstring the possible target space dimensions were seen to be 3, 4, 6 and 10, but for other branes the target space dimensions will be seen to differ. By quantizing the superstring only the 10-dimensional theory remained as a consistent theory. See [66] for details.

Another interesting question in the NSR approach is what happens if we impose higher order supersymmetry. Here the only possibilities are $N = 2$ and $N = 4$ in order to keep the (super-)conformal symmetry. These theories turn out to be quite uninteresting from a physical point of view due to their uninteresting critical dimensions (see [66] for details).

6

M-Theory

Here will be brought a brief review of the collecting theory dubbed M-theory. The letter M stands for something like "mother" or "membrane". The reason for why it is sometime referred to as membrane theory is that it in the low energy limit looks like $D = 11$ supergravity. So what is then M-theory? The historical origin lies in the discovery of various dualities in string theory in the mid 90's, see [69]. The idea to these came from the weak-strong duality in various field theories as previous chapters discuss. In string theory various discoveries lead to the conclusion that the five consistent perturbative string theories no longer should be seen as individual different theories but instead as if all were originating from the same theory which should be called M-theory. The reason for conjecturing this is of course the different dualities discovered which beside the string theories also contain $D = 11$ supergravity. As was seen in previous chapters all the string theories can be seen to live in ten dimensions with supergravity as their low energy backgrounds. Low energy in that it is the lowest order terms in α'. The basic procedure in studying duality properties is Kaluza–Klein reduction. We will see that these reductions will split the field representations of a higher-dimensional theory into new irreducible representations in the lower-dimensional theories and upon this see that different string theories and eleven-dimensional supergravity will map onto new equivalent theories in lower dimensions. The probes for this procedure are basic solutions to the low energy supergravity theories, called p-branes. The reason for their name is that they generalize the string, which would be a 1-brane, to higher-dimensional objects. The membrane would be a 2-branes while the point particle would be a 0-brane. These objects will be studied in next section.

For more detailed reviews on the subject the reader is referred to Refs. [70, 69, 71, 72, 73].

6.1 p-Branes

Here will be focused on some classical solutions to the equations of motion of the low energy supergravity theories, called p-branes. p-branes are p-dimensional objects propagating in time, and thus generalize the introduction of strings as 1-dimensional objects to that of arbitrary higher-dimensional objects. It might seem intuitive to introduce all types of higher dimensional objects, and not only strings, upon leaving the old point of view that particles should be pointlike objects. It must be stressed though, that the p-branes encountered here do not stand entirely on equal footing with the strings because they are only classical solutions to the low energy effective actions derived from string theory and actually only up to zeroth order in α'. There are thus two immediate questions one might ask, namely what if one started with p-branes as fundamental objects, what theories would one get and which would their critical dimensions be? The second is will these p-brane solutions exist at the non-perturbative level when taking in account for all orders in α'? The answers to these questions are yet not definite but the last years have given us a very good insight in the questions and although the answers only lie at the conjectural level, the belief concerning the last question is that the picture is correct to all orders. As has been seen the superstrings can only propagate in 10 dimensions as fundamental objects at the quantum level. In the classical regime, though, they can exist in 3, 4, 6 and 10 dimensions. In next subsection it will be clear that the classical dimensions for the membrane are instead 4, 5, 7 and 11 and although there exist no quantum theory for the membrane yet it would not be to surprising if this quantum theory eventually picked out only one of these dimensions and then preferably 11 which is the dimension of M-theory.

As the p-brane solutions are extreme supergravity black holes and therefore BPS states which breaks half the supersymmetry the first acquaintance would preferably be through ordinary black holes. In next subsection the concepts of black holes will be reviewed, and standard quantities such as inner and outer horizon, surface gravity, temperature and entropy will be introduced. These concepts follow all types of p-brane solutions. In this subsection the Reissner–Nordström black hole will be thoroughly presented and typically its extreme version where the mass is equal to the charge. This black hole is charged under a $U(1)$ vector field and would in the context be a 0-brane. In next subsection this will be generalized when the p-branes will be seen to be charged under a $U(1)$ $(p+1)$-form instead. Then the focus will be put on the brane scan which is where the possible existing branes in the various theories is derived. This will be solidly based on supersymmetry and dimensional analysis and is not a full proof of their existence, but today all types of solutions have been found.

6.1.1 Black holes

In four-dimensional Einstein gravity, black hole solutions have thoroughly been studied in the 60's and 70's, but the first classical solutions are already from 1916 when Schwarzschild first found his well known spherically symmetric one-parameter solution. Later in 1916 Reissner found a two parameter solution of a charged black hole. This was also independently discovered by Nordström in 1918 and is today known as the Reissner-Nordström black hole. It was not until 1963 that Kerr found yet another two-parameter solution of a rotating black hole which was generalized by Newman *et al.* in 1965 to build a complete three-parameter solution of a charged, rotating black hole. All these solution are found in the $D = 4$ dimensional Einstein-Maxwell theory given by the action

$$S_{EM} = \int d^4x\sqrt{-g}(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}) \tag{6.1}$$

The explicit three parameter solution can be found in [74] and is omitted because it is a bit out of the context of this subsection. More interesting is the thermodynamic theory of black holes which was mostly due to Bekenstein and Hawking, who saw that quantum mechanically black holes were not that black after all. To see this first some attention is put to what relevant quantities a black hole possess. First of all the three parameters of the Newman *et al.* solution is best seen as the mass, M, the angular momentum, J, and the charge, Q. In studying a black hole's thermodynamic properties there are three characteristics of a black hole that are of great importance, namely the *surface gravity*, κ, (not to be confused with the Einstein constant κ given by $2\kappa^2 = 16\pi G$), the angular velocity, Ω, and the electric potential at the event horizon, Φ. These are of course all derivable from the metric and the electromagnetic vector potential given by the three-parameter solution. They can be found to be

$$\kappa = \frac{4\pi}{A}(r_+ - M) \tag{6.2}$$

$$\Omega = \frac{4\pi}{A}\frac{J}{M} \tag{6.3}$$

$$\Phi = \frac{4\pi}{A}Qr_+ \tag{6.4}$$

where A is the area of the horizon given by

$$A = 4\pi(r_+^2 + \frac{J^2}{M^2}) \tag{6.5}$$

and the inner and outer horizon radii are

$$r_\pm = M \pm (M^2 - \frac{J^2}{M^2} - Q^2)^{1/2} \tag{6.6}$$

The surface gravity measures the force by which a unit test mass at the horizon must be exerted with at infinity in order to stay in place. This force is of course redshifted all the way out to infinity so the force needed at the horizon is infinite. Now a no hair theorem states that the only information contained in a black hole could be that of the three quantities, M, J, Q, so two black holes with the same mass, angular momentum and charge would be indistinguishable from each other even if one was formed by a collapsing star and the other by a giant imploding submarine. Now Bekenstein noted that there was a relation between two nearby solutions

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ \tag{6.7}$$

which looked very similar to the first law of thermodynamics

$$dU = TdS - pdV \tag{6.8}$$

This led Bekenstein to suggest that the black hole had a temperature given by some factor times the surface gravity and that the black hole had an entropy proportional to the area. There was a paradox before the discovery of a black hole's entropy because the second law of thermodynamics which states that the entropy always increases in a closed system did not hold in a theory with gravity. The classical area law theorem of a black hole states that the area always increases and thus opens for the possibility of the black hole to reduce the entropy of the outer region. This would not be compatible with the second law of thermodynamics. Now if one assigns an entropy to the black hole which is proportional to the area of its horizon, the total entropy, given by the entropy of the black hole plus the entropy of the outer region, would again obey the second law. In 1974 Hawking derived the temperature by a semi-classical analysis of the emission of particles near a black hole and found it to be

$$T = \frac{\kappa}{2\pi} \tag{6.9}$$

This should be compared to the Unruh effect found in 1976. Here Unruh showed that an accelerating observer in ordinary Minkowski space time feels that he is in a thermal bath with temperature proportional to the acceleration

$$T = \frac{a}{2\pi} \tag{6.10}$$

Again this is on par with the principle of equivalence. It can also be shown that a free falling observer in a black hole background (as, indeed, in any background) does not feel any temperature. Another interesting feature that arose through these calculations was that the temperature was independent of the number of different particles created. Following Bekenstein's reasoning this leads to the entropy of the black hole

$$S = \frac{A}{4} \tag{6.11}$$

A couple of implications follow from this discovery. The first is of course that a black hole does indeed have hair, there must be some microstructure that holds information and gives rise to this entropy. Secondly, this entropy law says that entropy is increased by clumping in a theory with gravity rather than as in non-gravitating systems where particles with a uniform distribution gives the highest entropy. The third observation is that black holes have negative specific heat which makes it unstable in a heath bath, either it burns up or cools down to zero.

Taking a look at the expression for the surface gravity one sees that there are possible configurations where the surface gravity vanishes even with the parameters different from zero. These are the so called extreme black holes. We will take a closer look at the Reissner-Nordström black hole which means that we put the angular momentum to zero. Here the solutions can be put in a more pleasant form and metric is

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2 d\Omega^2 \tag{6.12}$$

and the electro-magnetic vector potential reads

$$A = \frac{Q}{r}dt. \tag{6.13}$$

The introduced parameter Δ is given in terms of M and Q by

$$\Delta = r^2 - 2Mr + Q^2 = (r - r_+)(r - r_-) \tag{6.14}$$

with the inner and outer radii now taking the simple form

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{6.15}$$

For these horizons to exist we get a lower bound for the mass

$$M^2 \geq Q^2 \tag{6.16}$$

which look similar to the Bogomol'nyi bound seen in different field theories earlier on. The extreme Reissner–Nordström solution is then found simply by taking $M =\mid Q \mid$ and the metric reduces to

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2. \tag{6.17}$$

The extreme Reissner–Nordström metric is of course asymptotically Minkowski, stationary and spherically symmetric. To determine the asymptotic metric in the near horizon limit, $r \to M$, it is convenient to introduce a new radial coordinate R given by $MR = \left(1 - \frac{M}{r}\right)$. Taking only into count the leading terms in R the near-horizon metric of the extreme Reissner–Nordström black hole looks like the Robinson–Bertotti metric

$$ds^2 \sim M^2 \left(-R^2 dt^2 + \frac{dR^2}{R^2}\right) + M^2 d\Omega_2^2. \tag{6.18}$$

This is nothing but the metric of a two dimensional anti-de Sitter manifold times a two-sphere, $AdS_2 \times S^2$, so we see that the solution interpolates between a Minkowski vacuum and $AdS_2 \times S^2$. This is the first case of what can be seen as a 0-brane which is BPS in that that it satisfies the lower bound of the mass. When looking at the Reissner–Nordström black hole as a solution to $N = 2$ supergravity in $D = 4$, we can see this BPS state as a soliton. From the relation between the surface gravity and the Hawking temperature it is clear that these BPS black holes have zero temperature and are thus stable.

6.1.2 Typical p-brane solutions

In the various compactification schemes appearing in the Kaluza–Klein reductions of the supergravity theories at hand there are a lot of fields appearing. Although this severe complexity in the ansatz for a p-brane solution one can put most of these fields to zero and just look at one scalar field together with one anti-symmetric tensor field. The action describing the dynamics of these fields can in the Einstein frame be put to look like

$$S_D[g,\phi,A] = \int d^Dx\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2(d+1)!}e^{-\alpha\phi}F^2_{(d+1)}\right) \tag{6.19}$$

where $F_{(d+1)} = dA_{(d)}$ is the field strength of a d-form potential $A_{(d)}$, ϕ is the dilaton field and R is the scalar curvature. $d = p + 1$ will be the total dimension of the $p-$brane including the time direction. Furthermore, $\alpha = \alpha(d, D)$ is a numerical constant which depends on the dimension of both the space and the brane. For a single field p-brane it takes the value

$$\alpha^2 = 4 - \frac{2d\tilde{d}}{D-2} \tag{6.20}$$

where the notation $\tilde{d} = D - d - 2$ for what will be called the dual worldvolume dimension. If one rewrites the F^2 term as $F \wedge *F$ instead it is clear that the choice of potential is not that obvious. One could equally choose a potential to the dual field, i.e. $*F_{(\tilde{d}+1)} = d\tilde{A}_{\tilde{d}}$. This ambiguity is the reason for the p-branes to come in pairs. There is of course a great resemblance with ordinary gauge theory where we saw how magnetic monopoles appeared in a dual fashion to the electric monopoles. Here the magnetic monopoles were solitonic objects while the electric monopoles were fundamental objects. This will turn out to be similar in some cases with p-branes where one object will be of fundamental nature and therefore referred to as electrically charged while the dual will be solitonic and referred to as magnetically charged. (D-branes which are all solitonic still come in pairs but the role of what is fundamental and what is not is blurred.) The explicit electric and magnetic p-brane solutions to the equations of motion derived from the above action can be written

$$
\begin{aligned}
g &= \Delta^{-\frac{\tilde{d}}{D-2}} dx^2 + \Delta^{\frac{d}{D-2}} dy^2 && (6.21)\\
e^{\phi} &= \Delta^{\pm\frac{\alpha}{2}} && (6.22)\\
F &= \begin{cases} \epsilon \wedge d\Delta^{-1} & \text{electric} \\ -i_{d\Delta}\epsilon' & \text{magnetic} \end{cases} && (6.23)
\end{aligned}
$$

where the harmonic function $\Delta = \Delta(r)$ with $r := \sqrt{\delta_{m'n'} y^{m'} y^{n'}}$ is given by

$$\Delta = 1 + \left(\frac{a}{r}\right)^{\tilde{d}} \tag{6.24}$$

Characteristic for these p-brane solutions is that they break the P_D symmetry down to $P_d \times SO(D-d)$. Recall that this is nothing but the characteristics of an almost product structure. By letting $I = d - d'$ we have a suitable almost product structure compatible with the symmetry breaking of the brane solution. Now the Nijenhuis tensor of this APS vanishes which tells us that both the brane and the complementary distributions are integrable. The consequence of this is that the brane in this case can be seen as a foliation instead of merely an embedding. There have recently come several papers on rotating branes [75, 76, 77, 78, 79, 80, 81] where it is clear that the solutions again are compatible with an APS but where the brane no longer is integrable. From these solution and by just looking at the general splitting of the curvature components due to an APS it is clear that gauge field charged black holes in the dimensionally reduced space will correspond to rotating branes in the total space. When it comes to black holes charged under anti-symmetric tensor fields one can make the conjecture that these correspond to non-integrable distributions in superspace. See paper IV and V for the structure.

From the equations of motions for the anti-symmetric tensor field and the Bianchi identity

$$
\begin{aligned}
d * (e^{-a\phi} F) &= *J && (6.25)\\
dF &= *K && (6.26)
\end{aligned}
$$

where $J_{(d)}$ is a d-form current associated with the p-brane source, $K_{(\tilde{d})}$ is the $\tilde{d}$-form "magnetic" current. The electrically charged p-brane in D dimensions can be encircled by a $(D-d-1)$-dimensional sphere. It carries a conserved Noether charge

$$q_e = \int_{\mathscr{M}_{(D-d)}} *J_{(d)} = \int_{S^{D-d-1}} e^{-\alpha\phi} * F_{(d+1)} \tag{6.27}$$

As the p-branes solutions indeed are BPS states as they break half the supersymmetry[1] which of course not is clear from the above treatment. Their ADM mass per

[1] To be more precise they do in fact break a lot more than half the supersymmetry. They break the local supersymmetry of the supergravity theory down to a global supersymmetry with half the number of supersymmetry generators.

unit p-volume is given by

$$M_d \geq |q_e| e^{\alpha\phi_0/2} \tag{6.28}$$

for the electrically charged p-brane. ϕ_0 denotes the vacuum expectation value of the scalar field. In the dual picture the solitonic $\tilde{p}$-brane can be encircled by a $(d+1)$-dimensional sphere and carries a magnetic charge which is of topologically origin

$$q_m = \int_{\mathscr{M}(d+2)} *K_{(\tilde{d})} = \int_{S^{d+1}} F_{(d+1)} \tag{6.29}$$

Magnetic charges can be supported without sources at the core, so the solitonic $\tilde{p}$-branes are solutions to the equations of motion of the effective action $S_D(d)$ alone. The solitonic $\tilde{p}$-brane also preserves one-half of the underlying supersymmetry, but saturates another type of bound for the ADM mass per unit $\tilde{p}$-volume

$$M_{\tilde{d}} \geq |q_m| e^{-a(d)\phi_0/2} \tag{6.30}$$

Here one should notice the difference in sign which in string theories where the string coupling constant depends on the dilaton field in a similar fashion tells us that the mass of the magnetically charged solitonic p-branes depends on the inverse of the coupling constant to some power. This is what makes them lie outside the perturbative spectrum as their masses blow up as one does perturbations in a small coupling constant.

Before proceeding with the actual brane scan the membrane and the 5-brane in $D = 11$ supergravity will be investigated as an example of two dual pairs of p-branes discussed above. Here the treatment is even a bit easier because there exist no scalar field in the eleven dimensional supergravity theory. Here will also be seen that the p-brane solutions can be seen as an interpolation between two different topologically inequivalent vacua of the supergravity theory. The interpolation will be seen to between an anti-de Sitter space times a hypersphere and ordinary Minkowski space.

- **The fundamental M2-brane**
  From the general solution we can just read off the solution of the M-theory membrane by identifying $D = 11, d = 3, \tilde{d} = 6$

  $$ds^2 = \Delta^{-\frac{2}{3}} dx^2 + \Delta^{\frac{1}{3}} dy^2 \tag{6.31}$$

  with

  $$\Delta = 1 + \left(\frac{a}{r}\right)^6 \tag{6.32}$$

  The potential reads

  $$A_{(3)} = \pm\frac{1}{3!}\Delta^{-1}\epsilon_{mnp} dx^m \wedge dx^n \wedge dx^p \tag{6.33}$$

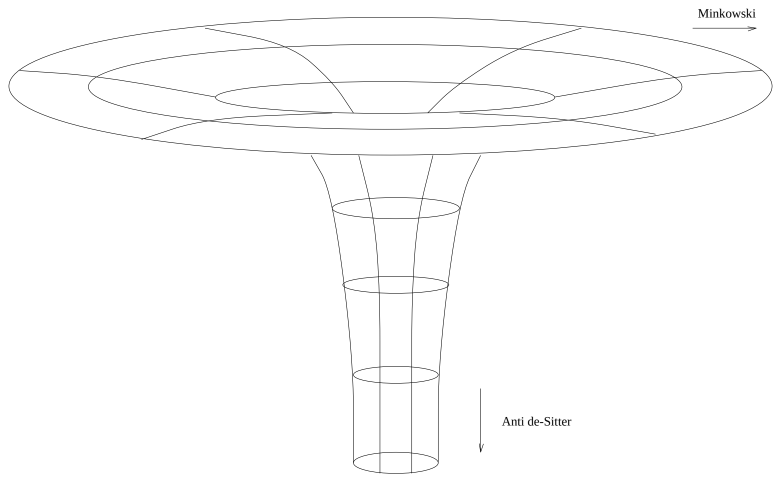

Figure 6.1: The Vacuum Interpolation of the Supermembrane Geometry.

It is now interesting to look at the coordinates of this solution picked from the general above. It certainly looks as if the M2-brane metric was singular at $r = 0$. This singularity though, is nothing but a coordinate singularity and the hypersurface $r = 0$ is nothing but the event horizon of the brane. This can be seen by introducing a Schwarzschild-type coordinate ρ, defined by $r^6 = \rho^6 - a^6$. The solution now reads

$$ds^2 = \Lambda^{\frac{2}{3}} dx^2 + \Lambda^{-2} d\rho^2 + \rho^2 d\Omega_7^2 \tag{6.34}$$

whith

$$\Lambda = 1 - \left(\frac{a}{r}\right)^6 \tag{6.35}$$

and the potential reading $A_{(3)} = \pm\frac{1}{3!}\Lambda\epsilon_{mnp}dx^m \wedge dx^n \wedge dx^p$. The new Schwarzschild-type coordinate makes the hypersurface $\rho = a$ the horizon instead of $r = 0$ in the original case. To get the picture of the membrane as a solution interpolating between two topologically inequivalent vacua it is preferable to introduce yet another set of coordinates defined by

$$R^3 = 1 - \left(\frac{a}{\rho}\right)^6 \tag{6.36}$$

With this new radial coordinate the metric looks like

$$ds^2 = ds^2_{asym} + a^2 \left[\frac{1}{(1-R^3)^{1/3}} - 1\right] \left[4\frac{dR^2}{R^2} + d\Omega_7^2\right] \tag{6.37}$$

where

$$ds^2_{asym} = R^2 dx^2 + 4a^2 \frac{dR^2}{R^2} + a^2 d\Omega_7^2 \tag{6.38}$$

In this form the metric of the M2-brane reveals the interpolation between the two topologically inequivalent vacua. As $R \to 1$ which correspond to $\rho \to \infty$ which is at spatial infinity the solution becomes flat $D = 11$ Minkowski. In the other limit when $R \to 0$ which is at the horizon $\rho = a$ the M2-brane metric becomes instead the standard metric on $AdS_4 \times S^7$. It should be stressed that the horizon $R = 0$ is not a singularity but the real singularity is encountered at the origin $\rho = 0$. It is here the embedding of the membrane can be put as a source term.

- **The solitonic M5-brane**

The M5-brane solution follows the standard prescription of p-brane solutions as the M2-brane did. The explicit solution can be read off the from the general solution now with $D = 11, d = 6, \tilde{d} = 3$

$$ds^2 = \Delta^{-\frac{1}{3}} dx^\mu dx^\nu \eta_{\mu\nu} + \Delta^{\frac{2}{3}} dy^m dy^n \delta_{mn} \tag{6.39}$$

with

$$\Delta = 1 + \left(\frac{a}{r}\right)^3 \tag{6.40}$$

The field strength taken in component form reads

$$H_{(4)} = \pm \frac{1}{4!} (\partial_{m'} \Delta) \delta^{m'n'} \epsilon_{n'p'q'r's'} dy^{p'} \wedge dy^{q'} \wedge dy^{r'} \wedge dy^{s'} \tag{6.41}$$

In a similar procedure as for the membrane we introduce an interpolating coordinate R, now defined by

$$r = \frac{aR^2}{(1 - R^6)^{1/3}} \tag{6.42}$$

After this substitution the M5-brane metric reads

$$ds^2 = R^2 dx^\mu dx^\nu + a^2 \left[\frac{4R^{-2}}{(1 - R^6)^{8/3}} dR^2 + \frac{1}{(1 - R^6)^{2/3}} d\Omega_4^2 \right] \tag{6.43}$$

The M5-brane can now be seen to interpolate between flat $D = 11$ Minkowski space as $R \to 1$, and $AdS_7 \times S^4$ as one approaches the horizon. So we have seen that both the fundamental M2-brane and the solitonic M5-brane geometry interpolates between two of the vacua of eleven-dimensional supergravity. This picture is illustrated in 6.1. Interesting to see is that this metric is completely symmetric under the discrete isometry $R \to -R$. One can therefore analytically continue the exterior metric through the horizon, at $R = 0$, and go over to negative values of R. But then the geometry will look completely the same on the inside as on the outside. There is thus no singularity behind the horizon in the M5-brane case but the solution is completely solitonic.

6.1.3 The Brane-scan

The brane scan consists of an analysis of the possible presence of different kinds of p-branes in the various supergravity theories. It is solidly based on the analysis of supersymmetry properties in different dimensions. There are four different types of branes, these are the scalar, vector, tensor and gravitational branes. These will here be referred to as Sp-, Dp-, Tp- and Gp-branes repectively[2]. The brane scan is a classification of what branes can exist in what dimensions and in what theories in particular. It is again convenient to introduce $d = p + 1$ as the dimension of the embedded p-brane in space-time dimension D. A list of all acceptable branes except for the gravitational ones can be found in Fig. 6.2. These refer to which type of multiplets are to be embedded into target space. In Fig. 6.2 account has only been taken to in what possible dimensions there is a match between the number of fermions and the number of bosons. This does therefore only give us a necessary condition but not a sufficient condition for their existence. The next step is to look at the supersymmetry algebra and see what possible central charges there are. Finally one must of course show that the explicit solutions exist through the equations of motion but today we know that all compatible with the supersymmetry algebra do exist. But first let us take a look at how to get the brane scan.

- **Sp-branes**
  To match the bosonic and the fermionic degrees of freedom in the scalar multiplet case we first note that the complementary dimension of the embedding gives the number of scalar degrees of freedom and that κ-symmetry reduces the fermionic degrees of freedom by half which the fermionic equations of motion also does. To match these we are left with the equation

$$D - d = \frac{1}{4}MN \tag{6.44}$$

  where M is the irreducible spinor dimension in D-dimensional Minkowski space, and N is the number of supersymmetries. M can be read off from Table 4.1 and N is restricted to $N = 1$ in all cases but the string case where the independence of the left and right moving sector gives us a possibility of not only $N = 1$ but also $N = 2$.

- **Dp-branes**
  This is the case when the p-brane is represented by a vector multiplet. Here we again have $D - d$ scalars but here is also a world-sheet vector with $d - 2$ degrees of freedom and the imposed equality becomes instead

$$D - 2 = \frac{1}{4}MN \tag{6.45}$$

[2]These categories refer to a fixed dimensionality. Upon dimensional reduction and compactification these categories can mix.

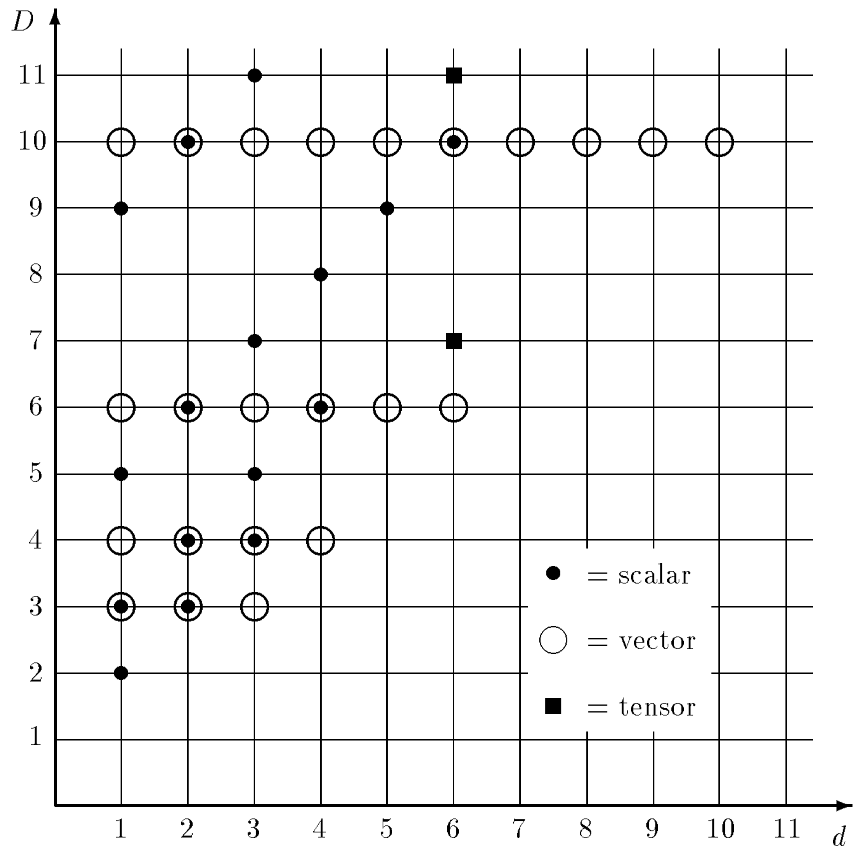

Figure 6.2: The brane-scan.

Here we see that we have no restriction on the value of d but only on the dimension of the target space which will coincide with those for which we have possible string theories, due to the number 2 in the equation above, namely $D = 3,\ 4,\ 6,\ 10$.

- **Tp-branes**
  The Tp-branes are represented by a world-sheet tensor multiplet. As before there are $D - d$ scalars but now also bosonic degrees of freedom from a tensor field. The question is what tensor field there can be. This question can be solved by looking at what possible tensor multiplets there exist in the representation table (4.4). Here it is clear that there is only one possibility and that is for $d = 6$ which means that we are restricted to a 5-brane. The degrees of freedom for the self dual tensor in 6 dimensions are 3 and the imposed equality turns out to be

$$D - 3 = \frac{1}{4}MN \tag{6.46}$$

So the possible T5-branes can exist in the same dimensions as the membrane though we are of course restricted to dimensions $D > 6$. This leaves us with the only possibilities $D = 7,\ 11$.

- **Gp-branes**
The possible Gp-branes was not illustrated in the picture but will nevertheless be listed here. The branes in the different theories are in correlation to the existence of p-form central charge in the supersymmetry algebra. But what is characteristic for all these algebras is the existence of the momentum operator P which can be shown to represent a type of string called the pp-wave. Its dual is a $(D-5)$-brane which in fact is a Kaluza-Klein monopole. So for $D \geq 5$ there are two new types of branes referred to as gravitational branes due to carrying the electric and magnetic charge of the graviphoton respectively. See [82].

These are the possible branes from a dimensional point of view, (actually there can be some other 9-branes, see [82]), but the next step is now to look at the presence of their respective central charges in the supersymmetry algebra[83]. Again the different theories at hand are the various supergravity theories which represent the low energy effective actions of string theory and M-theory.

- **D=11 SUGR**
The supersymmetry algebra in ordinary $D = 11$ supergravity takes the form

$$\{Q_\alpha, Q_\beta\} = (\Gamma^a)_{\alpha\beta} P_a + (\Gamma^{ab})_{\alpha\beta} Z_{ab} + (\Gamma^{abcde})_{\alpha\beta} Z_{abcde} \tag{6.47}$$

From the algebra it follows that there exist one membrane and one 5-brane in the spectrum. These are of course the fundamental M2-brane and the solitonic M5-brane discussed earlier see Refs. [84, 85]. From the brane scan one finds that the M2-brane is represented by a scalar multiplet while the M5-brane is represented by a tensor multiplet. From the algebra one can also track down the gravitational branes which are $G1$-brane and its dual $G6$-brane. (There is actually an extra 9-brane conjectured in [82] but it is not present as a central charge in the algebra.)

- **Type IIA**
In the type IIA supergravity the supersymmetry algebra is just the dimensional reduction of $D = 11$ SUGR to $D = 10$, and looks like

$$\begin{aligned}\{Q_\alpha, Q_\beta\} \;=\; & (\Gamma^a)_{\alpha\beta} P_a + (\Gamma^{11})_{\alpha\beta} Z + (\Gamma^a \Gamma^{11})_{\alpha\beta} Z_a + (\Gamma^{ab})_{\alpha\beta} Z_{ab} + \\ & + (\Gamma^{abcd}\Gamma^{11})_{\alpha\beta} Z_{abcd} + (\Gamma^{abcde})_{\alpha\beta} Z_{abcde}\end{aligned} \tag{6.48}$$

Here we of course read off the fundamental string and its dual the solitonic S5-brane. From the algebra together with the brane scan one also finds Dirichlet branes of the types D$0, 2, 4, 6, 8$ found in Refs. [86, 87, 88]. Here the gravitational brane $G1$ associated with the momentum generator has its dual in the $G5$-brane. (There is an extra 9-brane in this case too as will be the case for all the ten dimensional supergravity theories.)

- **Type IIB**
  The supersymmetry algebra of type IIB reads

$$\begin{aligned}\{Q^I_\alpha, Q^J_\beta\} &= \delta^{IJ}(\Gamma^a\mathcal{P})_{\alpha\beta}P_a + (\Gamma^a\mathcal{P})_{\alpha\beta}\tilde{Z}^{IJ}_a + \epsilon^{IJ}(\Gamma^{abc}\mathcal{P})_{\alpha\beta}Z_{abc} + \\ &\quad +\delta^{IJ}(\Gamma^{abcde}\mathcal{P})_{\alpha\beta}Z^+_{abcde} + (\Gamma^{abcde}\mathcal{P})_{\alpha\beta}\tilde{Z}^{+IJ}_{abcde} \qquad (6.49)\end{aligned}$$

  where $\mathcal{P}$ is a chiral projector. Beside the fundamental string and its solitonic dual the S5-brane, there also exists Dirichlet branes in type IIB but now with odd values namely D$-1, 1, 3, 5, 7, 9$-branes see Refs. [86, 87, 89, 88]. Here is also the $G1$, its dual the $G5$-brane plus the extra 9 brane.

- **Type I/Het**
  These theories are a bit poorer because of the absence of the Ramond-Ramond fields. Their supersymmetry algebra takes the much simpler form

$$\{Q_\alpha, Q_\beta\} = (\Gamma^a\mathcal{P})_{\alpha\beta} + (\Gamma^{abcde}\mathcal{P})_{\alpha\beta}(Z^+)_{abcde} \qquad (6.50)$$

  $\mathcal{P}$ is again a chiral projector. Although not manifest there exist a D1-brane beside the fundamental string and its solitonic dual the S5-brane see Refs. [90, 87, 91, 92]. Even in this case there are the $G1$, the $G5$-brane and the extra 9 brane.

While studying the brane spectrum it was clear that there was basically four types of branes namely the Sp-, Dp-, Tp- and Gp-branes. The next couple of subsections will look at the basic differences between these branes and what their dynamical properties are.

6.1.4 Sp-Branes

The dynamics of Sp-branes follows the bosonic principle of minimal surfaces. There is a slight problem with this principle in the supersymmetric case though. Although everything can be made manifestly target space supersymmetric, by simply formulating it in superspace, it is a bit harder to exploit the world-sheet supersymmetry and thereby get the right degrees of freedon of the scalar multiplet. The trick that resolves this problem is the concept of κ-symmetry. Actually there are three different approaches in order to address this problem, one is the case discussed where the world-sheet is embedded into target superspace. One could equally look at world-sheet superspace embedded into target space or make the so called doubly supersymmetric ansatz and take world-sheet superspace and embed it into target superspace. The latter has been used with great success in the T5-brane cases where there is no action describing the dynamics of the brane due to the self-dual tensor field. Though today one must say that the doubly supersymmetric ansatz is the most beautiful in a geometrical perspective we will follow the standard κ-symmetric ansatz as this was the first to reveal its secrets.

Start with the Dirac-Nambu-Goto action for the Sp-brane, i.e.

$$I_{DNG} = -T_p \int \sqrt{-\det g} \tag{6.51}$$

and let the metric be induced from the bosonic metric of target superspace, that is $g_{mn} = \underline{E}_m{}^{\underline{a}} \underline{E}_n{}^{\underline{b}} \eta_{\underline{ab}}$. It can be seen that this action is not world-sheet supersymmetric unless one adds an extra term known as a Wess-Zumino term. This term is the pullback of the anti-symmetric tensorfield under which the brane is charged. Put together the action becomes supersymmetric with half the supersymmetry of the target space. In fact the Wess-Zumino term makes the action κ-symmetric and the desired supersymmetry is reached upon gauge-fixing this κ-symmetry. The total action including the Wess-Zumino term takes the form

$$I = I_{DNG} + I_{WZ} = -T_p \int \sqrt{-\det g} + T_p \int f^* B_{(p)} \tag{6.52}$$

It is invariant under local κ-transformations of the form

$$(\delta_\kappa Z^{\underline{M}}) \underline{E}_{\underline{M}}{}^{\underline{a}} = 0, \quad (\delta_\kappa Z^{\underline{M}}) \underline{E}_{\underline{M}}{}^{\underline{\alpha}} = \kappa^{\underline{\alpha}} \tag{6.53}$$

The fact that the κ-symmetry breaks half the target space supersymmetry, or rather preserves half of the supersymmetry, makes these p-branes BPS states so they will satisfy the Bogomol'nyi bound. Their masses are thus completely given by the charges,

$$M^2 = q_e^2 + q_m^2 \tag{6.54}$$

where the charges are given by

$$q_e = \int_{S^{D-p-2}} *F_{(p+2)} \tag{6.55}$$

and

$$q_m = \int_{S^{p+2}} F_{(p+2)} \tag{6.56}$$

The above mass formula holds for any dyonic states. The charges must satisfy the generalized Dirac quantization condition [93, 94]

$$q_e q_m = 2\pi n, \quad n \in \mathbb{Z} \tag{6.57}$$

As usual the electric charges are charged under the anti-symmetric tensor field and must in fact be inserted as source fields while the magnetic charges are of topological origin and are therefore solitonic. Now both solutions need to satisfy the Bogomol'nyi bound and must therefore have finite energy, and in that sense be solitonic. This is no problem though, because all anti-symmetric tensor fields come in pairs with their duals and what is a electric solution for one field is the magnetic for the dual field. An important thing to notice is that certain Sp-brane configurations in fact bring the target space supergravity theory on-shell in order for the action to be κ-symmetric. This is the case for the S2-brane in eleven dimensional supergravity and a lot of things suggest that this should rather be the case for all branes. The origin of how the branes put constraint on the target space is basically that the κ-symmetry is not fulfilled unless there are certain conditions on the target space torsion and antisymmetric tensor fields as they enter in the variation of the action. Now these condition can be so strong as to in fact bring the whole background theory on-shell. This is the case for the membrane and also the $T5$-brane in eleven dimensions.

6.1.5 Dp-Branes

This subsection will be very brief and only give some major results in the theory regarding D-branes and refer to [95, 88, 96, 97, 98, 99, 100, 101, 102] for a deeper study of the subject. As was clear from the brane scan where the classification table of Dp-branes are referred to as to the branes with a vector multiplet. The D stands of course for Dirichlet and is due to the fact that D-branes are objects on which open strings can end. This is seen when T-dualizing the open string where one finds that the endpoints are stuck to certain hypersurfaces. The dimension of these hypersurfaces is equal the number of compactified dimension of the T-dualization. These surfaces come to be called D-branes because they are in fact the same as those branes with a vector multiplet living on it. Now these branes are dynamical objects, whose motions can be described with the following action

$$I_{DBI} = -T_p \int e^{-\phi} \sqrt{-\det(g + \mathcal{F})} \tag{6.58}$$

where g is the pullback metric and $\mathcal{F} = 2\pi F - f^* B$ and $F = dA$. A is of course the vectorfield in the multiplet of the brane. The equations of motions was originally derived using the usual technique of requiring the β-functionals to vanish but now with an additional boundary term to the original string action action. This boundary term looks like

$$\oint_{\partial\Sigma} d\tau (A_m \partial_\tau X^m + \phi_{m'} \partial_n X^{m'}) \tag{6.59}$$

In addition to the ordinary β-functional calculations for the metric, the dilaton and the anti-symmetric tensorfield one must here impose the vanishing of the β-functional with respect to A and ϕ. This lead to the equations of motion for the D-brane as well as those for the A field. These equations of motion can then be reproduced by varying the action above. Again of course to lowest order in α'.

As was seen in the brane-scan these Dp-branes were restricted to dimensions $D = 3, 4, 6, 10$ where of course D in this case stands for the dimension of the target space. So it is clear that they are existing in $D = 10$ which is the critical dimensions of the superstrings. In these theories they have come to play a big role in the understanding of non-perturbative effects. There are some basic properties of D-branes namely

- They are BPS-saturated solitons and therefore break half the supersymmetry.
- The tensions scale as g_s^{-1} and D-branes are therefore non-perturbative.
- They are charged under the RR fields and satisfy the Dirac quantization condition, $q_e q_m = 2\pi n$.

As the Dp-branes are BPS states we must require for the action to be κ-symmetric in order to preserve half the supersymmetry of the target space. But as was the case for the Sp-branes where an additional Wess-Zumino term had to be included in order to make the action κ-symmetric, the same is true for the Dp-branes. Now the action, although looking very similar to the original Sp-brane action, it also includes the vector field which characterizes the D-brane. The Wess-Zumino term will not be as simple as in the case with the Sp-branes because it must additionally include the field strength of the vector field. It has been shown [103] that the correct way of including the Wess Zumino term is in the form

$$I_{WZ} = T_p \int e^{\mathcal{F}} \wedge C \tag{6.60}$$

for the type IIA and type IIB theory with RR fields

$$\begin{cases} C_{(1)} + C_{(3)} + C_{(5)} + C_{(7)} + C_{(9)} & \text{(IIA)} \\ C_{(0)} + C_{(2)} + C_{(4)} + C_{(6)} + C_{(8)}. & \text{(IIB)} \end{cases} \tag{6.61}$$

which makes the total action look like

$$I = -T_p \int e^{-\phi} \sqrt{-\det(g + \mathcal{F})} + T_p \int e^{\mathcal{F}} \wedge C \tag{6.62}$$

This action is now κ-symmetric and we are left with only half the target space supersymmetry on the world sheet [104, 105, 106].

One of the most interesting cases in which these D-branes have played an important role is in calculating the entropy of black holes [107] where the microscopic structure of the black hole was calculated by counting D-brane states. Originally the entropy of a black hole due to Hawking and Bekenstein was only found through the structure of the differential mass equation after the temperature was derived using a semi-classical analysis of particle creations near a black hole. It is thus a remarkable feature to be able to re-derive the result by actually calculating the microscopic structure of the black hole.

6.1.6 T5-Branes

The treatment of the T5-brane is a bit different due to the existence of the self-dual 3-form, which makes it impossible to write down an action to describe the dynamics, without introducing auxiliary fields. For an action formulation containing auxiliary fields, see [108, 109]. There is another way of treating branes, though, namely that of the so called superembedding formalism or the doubly supersymmetric ansatz. In this formalism one makes yet another approach instead of those of NSR and GS. Here the brane is described by an superembedding, now from world-sheet superspace to target space superspace. The fermionic coordinates of the world-sheet are half in number to those of the target space. Formulated in this way there is manifestly both target space and world-sheet supersymmetry. Though here the problem is that the field content of the world-sheet is larger than that of the representation. Recall the representation for the T5-brane from Table 4.4

$$(1,1;1) \oplus (3,1;1) \oplus (2,1;2) = \phi \oplus A^{+}_{ab} \oplus \psi^{i}_{a} \tag{6.63}$$

As shown in [110, 111], the so called embedding condition which looks like

$$\mathscr{E}_{\alpha}{}^{\underline{a}} = 0 \tag{6.64}$$

will reduce the field content to exactly that of the representation above. Through an induced torsion equation this embedding condition put restrictions on the target space background. In fact in the 11-dimensional case it will put us on-shell. In the 7-dimensional case though, we have to put an extra constraint on the torsion tensor to go on-shell. This was done in paper II. It should be stressed that this embedding condition basically states that the fermionic part of the world-sheet is entirely embedded into the fermionic part of the target space. It is this condition that makes the world-sheet multiplet to reduce to the right number of degrees of freedom.

6.1.7 Gp-Branes

The structure of the Gp-branes is not as thoroughly understood as it is with the other branes so the discussion is omitted here. The interested reader might consult Hull [82] for a review of the concept. In that article he also describes the conjectured extra 9-branes appearing in the various supergravity theories.

6.2 Web of dualities

Today there is overwhelming evidence that all consistent string theories are indeed different flavors of the same fruit. The name of the game is string duality. Counting p-branes as non-perturbative probes of these dualities brings us conjectures of the truth of the dualities even at the non-perturbative level. Beside the five consistent superstring theories, the eleven-dimensional M-theory also arises as a candidate for the more fundamental non-perturbative theory. The problem is that basically the only thing known about M-theory is that it in the low energy limit looks like $D = 11$ supergravity. Regarding the string theories higher order terms of the background fields can be obtained through the β-functional calculation, but M-theory lacks that possibility. If the membrane, which probably serves as the fundamental object of M-theory, could be quantized in the future, the higher order terms of M-theory would probably be obtained in a similar way to those in the string theories. Today this is impossible and one has to rely on the higher order terms of string theory and require that M-theory ends up into these through compactification. The duality relating all these theories are purely on the conjectural level due to the fact that only the low energy effective actions are taken into account. Nevertheless there are severe evidence that the dualities should generalize to the full theories. The firm base to this evidence is supersymmetry. As noticed already in the previous chapter, the intricate structure of supersymmetry enforces itself through procedures such as embeddings. That is to say that a sub-theory that is supersymmetric must lie in a totally supersymmetric theory. The small amount of supersymmetry representations thus makes the number of possibilities very tiny. The brane-scan was an example to how the branes with their local supersymmetry properties can be embedded into a larger background theory with double the amount of supersymmetry. When it comes to dualities in string theory and M-theory the higher dimensional theories will reduce in such a way that the field representations of the respective theories map to some possible field representation in the lower-dimensional theory. As these representations are very restrictive so are the possible outcomes of these compactifications. The web of dualities is the result of these restrictive possibilities.

Here will be discussed the basic dualities involving M-theory, including the five superstring theories. The dualities are proved for the low energy effective actions which include, beside $D = 11$ supergravity and $D = 10$ supergravity theories, also $N = 2$ and $N = 1$ supergravity in D=9. The low energy effective actions of these theories are related by duality transformations seen in Fig. 6.3. There are three basic types of duality transformations, namely T-duality, S-duality and U-duality which will be described in short.

- **T-duality**

  The mass spectrum of string theory compactified on a circle looks for the first oscillator level like

$$M^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{mR}{\alpha'}\right)^2 \tag{6.65}$$

  The first term is due to the quantized Kaluza–Klein momentum modes while the second term corresponds to the winding number of the string around the circle. So the mass is invariant under the simultaneous exchange of the Kaluza-Klein momentum modes with the winding modes, $n \leftrightarrow m$, and letting $R \rightarrow$

α'/R. This is referred to as T-duality where the T originates from toroidal compactification. Two different theories are said to be T-dual if they are related through this type of transformation. As T-duality makes no non-trivial coupling between the different string coupling constants of the two theories it can be probed at the perturbative level. We say that T-duality is a weak-weak duality as it can be proved order by order in perturbation theory. At the non-perturbative level though, T-duality remains at the conjectural level. For a review on the subject see Refs. [112, 113].

- **S-duality**
  This duality is in contrast to T-duality a strong-weak duality [114] because it relates the string coupling constants of the different theories in a non-trivial manner. Typically the $\mathbb{Z}_2$ generated by $g_s \to g_s^{-1}$ exists as a sub-group of the total S-duality group. Therefore the strongly coupled regime of one theory is related to the weakly coupled regime of another through an S-duality transformation. This makes it impossible to prove order by order in some perturbation expansion. Probes for this theory must then be dressed in BPS states, which are topological objects and must be stable under the duality transformation. These BPS states are typically the p-branes discussed previously.

- **U-duality**
  Suppose we are relating some theories by duality transformations by first performing a toroidal compactification in which these theories are T-dual and then relate these by some other theory by another compactification by which they are S-dual, then the intuitive picture would be that the total duality group of this transformation would be the direct product of the two of them. Now Hull and Townsend have conjectured that not only this direct product becomes the duality group but in fact a enlargement of it. The conclusion is that by doing a compactification step by step you loose information about its original structure and in that sense miss some "duality regions". See Ref. [115] for a complete study of the conjecture.

As is seen in the duality web the severe restriction on possible supersymmetric theories makes the world more easier when coming down in dimensions. In $D = 9$ there are only two types of string theories left and are simply called type I and type II respectively. Here I and II stands for the amount of supersymmetry in the respective theory. If one compactifies further dimensions these will eventually also be equal if the supersymmetry is broken in such a way that both theories obtain the same number of supersymmetries. The number of preserved supersymmetries through compactification is due to the various holonomy groups of the respective internal manifolds. Going down to four dimensions all different possibilities are listed in Table 6.1. In the cases listed in the web Fig. 6.3 only one case will reduce the amount of supersymmetry and that is M-theory on an orbifold. This will be discussed in the sequel.

Manifold	$\mathscr{H}$	Preserved SUSY
$Generic_7$	$SO(7)$	0
$Joyce_7$	G_2	$\frac{1}{8}$
CY_6	$SU(3)$	$\frac{1}{4}$
HK_4	$SU(2)$	$\frac{1}{2}$
Torus	1	1

Table 6.1: CY stands for Calabi-Yau and HK for hyperkähler.

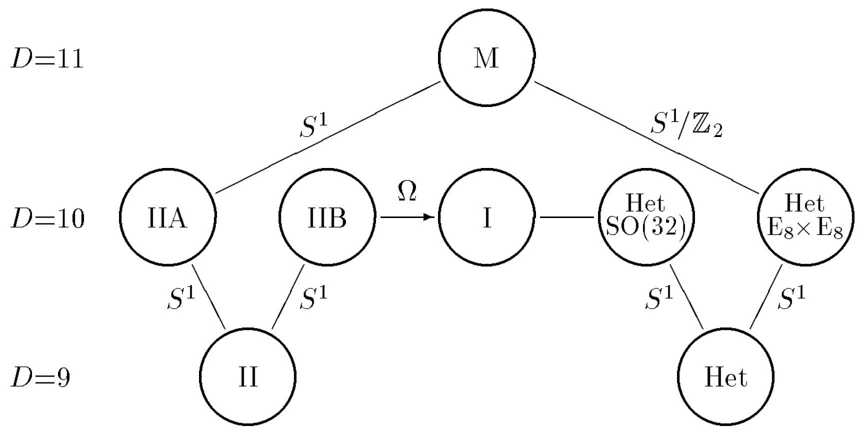

Figure 6.3: The web of dualities.

6.2.1 M-theory $\leftrightarrow$ Type IIA

The low energy effective action of M-theory is D=11 supergravity. So the duality between M-theory and type IIA string theory will is an S-duality in the sense that it relates the strong coupling limit of type IIA with the low energy limit of M-theory, namely D=11 supergravity. The duality transformation is obtained through compactifying D=11 supergravity over a circle, S^1. This will split the field representations into

$$\mathbf{44} \rightarrow \mathbf{35} \oplus \mathbf{8}_v \oplus \mathbf{1} \quad (6.66)$$

$$\mathbf{84} \rightarrow \mathbf{56}_v \oplus \mathbf{28} \quad (6.67)$$

$$\mathbf{128} \rightarrow \mathbf{56}_s \oplus \mathbf{8}_s \oplus \mathbf{56}_c \oplus \mathbf{8}_c \quad (6.68)$$

with the following relations between the bosonic fields

$$g = G^{10} + e^{2\gamma}(dy - A)^2 \quad (6.69)$$

$$A_{(3)} = C_{(3)} + B_{(2)} \quad (6.70)$$

Here the radius of the circle is $R_{11} = e^{\gamma}$. The low energy supergravity action of type IIA theory will be recovered by rescaling the metric like $g_{mn} = e^{\gamma} G_{mn}$ and through this process the string coupling constant, $g_s := e^{\phi}$, can be read off from the original type IIA supergravity action. The relation between the string coupling constant and the radius becomes

$$g_s = R_{11}^{3/2} \tag{6.71}$$

By compactification all type IIA branes are obtained from the much lesser spectrum of M-theory. These include the R-R-charged Dp-branes which scale as g_s^{-1} and the NS-NS charged five-brane which scales as g_s^{-2}. The complete perturbative type IIA string theory is obtained by taking the limit $R_{11} \to 0$. For a complete analysis see [69]

6.2.2 M-theory $\leftrightarrow$ Het $E_8 \times E_8$

These theories are also S-dual to each other. Here the $D = 11$ supergravity theory is compactified on an orbifold which is taken to be $S^1/\mathbb{Z}_2$, where the equivalence class is made by equating the two parities of the circle coordinate and is thus just a closed interval. This construction will break half the supersymmetry and leave us with the 16 supersymmetries of the heterotic string. The field representation truncates down to

$$\mathbf{44} \quad \to \quad \mathbf{35} \oplus \mathbf{1} \tag{6.72}$$

$$\mathbf{84} \quad \to \quad \mathbf{28} \tag{6.73}$$

$$\mathbf{128} \quad \to \quad \mathbf{8}_s \oplus \mathbf{56}_c \tag{6.74}$$

Here only the symmetric fields with respect to the parity transformation will survive. For more details see Hořava and Witten [116, 117].

6.2.3 Type IIB

In previous section containing various supergravity theories the type IIB supergravity action was written, (excluding the self-dual form), in a manifest self-dual form. The low energy type IIB theory have classically a $SL(2,\mathbb{R})$ duality symmetry. Enforcing Dirac quantization conditions this symmetry group to is reduced to its quantum representative, $SL(2,\mathbb{Z})$. The symmetry transformation acts like

$$M \to \Lambda M \Lambda^T \equiv \tau \to \frac{a\tau + b}{c\tau + d}; \quad \text{and} \quad \tilde{B} \to (\Lambda^T)^{-1}\tilde{B} \tag{6.75}$$

where

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{6.76}$$

as seen in the supergravity section. This must be referred to as an S-duality transformation as it includes the transformation $\phi \to -\phi$ and thus changes the regimes of strong and weak couplings. See [69] for more details.

6.2.4 Type IIA $\leftrightarrow$ Type IIB

This is the first example, so far, of two T-dual theories. Although type IIB is chiral while type IIA is non-chiral both will end up in the $D = 9$, $N = 2$ supergravity representation when compactified on a circle, S^1. In $D = 9$ there is no input about 10 dimensional chiralities so both theories will coincide and their field representations will end up in the same representation in 9 dimensions. In fact they have to do this because there is only one supergravity representation for $N = 2$ in $D = 9$. So the conclusion is that the two type II theories are equivalent in the perturbative regime in D=9 and a conjecture states that this should be the case also at the non-perturbative level, see [118]. The natural interpretation of this is that there is only one type II superstring theory below 10 dimensions and the split into type IIA and type IIB comes only into play while extending the compactification radii to infinity.

6.2.5 Het $SO(32)$ $\leftrightarrow$ Het $E_8 \times E_8$

These theories are also T-dual to each other. As was seen in the previous chapter while constructing the two heterotic string theories, the requirement for a consistent string theory was that the 16-dimensional lattice that defined the 16-dimensional torus, upon which the extra internal left moving degrees of freedom were compactified, had to be Euclidean and self-dual. Now there were only two of them which accounts for the two different types of heterotic strings. If the theories are compactified down to 9 dimensions or less, this even Euclidean self-dual lattice will be replaced by an even Lorentzian self-dual lattice $\Lambda_{16+d,d}$ instead, where d counts the dimension of the torus T^d upon which the theory is compactified. In this case it will turn out that there is in fact only one choice of lattice and we can draw the conclusion that there is only one heterotic string theory below 10 dimensions as was the case for the type II strings. The T-duality group becomes in this case that of $O(16 + d, d; \mathbb{Z})$, for more details read [119, 120, 121].

6.2.6 Type I $\leftrightarrow$ Het $SO(32)$

The conjecture here is that there is a strong-weak duality (S-duality) between these theories. The field transformations look like

$$\phi_I = -\phi_{Het}, \quad g_I = e^{-\phi_{Het}} g_{Het}, \quad C_{(2)} = B_{(2)} \tag{6.77}$$

and the duality group is thus $\mathbb{Z}_2$. The string coupling constant of the two theories are inversely related, i.e. $g_s \leftrightarrow g_s^{-1}$, which makes it a strong-weak duality. A detailed analysis can be found in [114].

6.3 Holography and the AdS/CFT conjecture

From a conjecture due to 't Hooft in 1993 the holographic principle named by Susskind in 1994 [122, 123] arose as a concept for gravitating systems. The holographic principle states that; *The fundamental degrees of freedom in a consistent quantum theory of gravity reside at the boundary and not in the interior of space-time. On the boundary there is precisely one degree of freedom per Planck area.* In a D dimensional manifold with minkowskian signature the area of the boundary refers to as the area of a $D-2$ dimensional hypersurface enclosing a $D-1$ dimensional volume of space. The total number of degrees of freedom inside a closed box is thus proportional to the area of the surrounding of the box. Although this principle today only exist at the conjectural level, there is some evidence in its favor. The fundamental origin of the holographic principle is through the study of black holes. It is evident that the degrees of freedom in some sense goes like the entropy of the system, and for a black hole the entropy is given through the Hawking–Bekenstein relation, stating that the entropy of a black hole is proportional to the area of its horizon. Furthermore from the no-hair theorem the information loss problem from the creation of a black hole can only be solved if the information is kept at the surface. The holographic principle could solve this if the information about the creation of the black hole is kept by the hologram of its surrounding surface.

At the end of 1997 Maldacena [124] found significant evidence for a specific theory obeying this holographic principle. It was through the study of extreme black holes in terms of D-branes that Maldacena made his break through. A basic feature of D-branes is that they have a local vector field living on the brane. This vector field is generically a $U(1)$ field but by stacking several D-branes on top of each other this $U(1)$ is extended to a $U(N) \sim U(1) \times SU(N)$ gauge field, where N denotes the number of stacked D-branes. The dynamics of the stacked D-branes is not yet entirely understood but in the low energy limit the gauge field is described by an ordinary Yang–Mills action. By studying the geometry surrounding N stacked $D3$-branes in type IIB superstring theory a lot of interesting things were found. First of all in the near horizon limit the space looked like $AdS_5 \times S^5$. Furthermore there are two dimensionless parameters both in the geometry of this black hole solution and in the gauge theory describing the dynamics of the vector field on the brane. In the black hole solution they are the string coupling constant, γ_s, and the common radius of the anti-de Sitter space and the sphere divided by the string length, R/l_s. In the gauge theory it is the number of colours, N, and the Yang–Mills coupling constant, g. This lead Maldacena to the conjecture that there should be a one-to-one correspondence between the physical degrees of freedom of type IIB string theory on $AdS_5 \times S^5$ and conformal $\mathcal{N} = 4$ supersymmetric Yang–Mills theory with gauge group $SU(N)$ on the boundary of the anti-de Sitter space. As the boundary of the anti-de Sitter space is ordinary Minkowski this conjectures relates a pure gauge theory on a Minkowski, space with a theory containing gravity in the bulk. Through this analysis the specific relation between the dimensionless parameters were obtained.

$$g_s = g^2, \qquad \frac{R}{l_s} = (g^2 N)^{1/4} \tag{6.78}$$

This conjecture has later been extended by others to include non-supersymmetric versions of Yang–Mills theory [125]. These are interesting examples that if true validate the holographic principle and can give us a better insight in the quantum theory of gravity.

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Notation index

$\underline{g}$	Target space metric
$\underline{E}^{\underline{A}}$	Target space vielbein
$\Omega_{\underline{A}}{}^{\underline{B}}$	Target space connection
$\underline{T}^{\underline{A}}$	Target space torsion
$\underline{R}_{\underline{A}}{}^{\underline{B}}$	Target space curvature tensor
$\underline{d}$	Target space exterior derivative
$\underline{\theta}$	Target space canonical one-form
$\underline{\nabla}$	Target space covariant derivative
g	Intrinsic metric
E^A	Intrinsic vielbein
$\Omega_A{}^B$	Intrinsic connection
T^A	Intrinsic torsion
$R_A{}^B$	Intrinsic curvature tensor
d	Intrinsic exterior derivative
$\mathcal{P}$	Intrinsic canonical one-form
∇	Intrinsic covariant derivative
g'	Normal metric
$E^{A'}$	Normal vielbein
$\Omega_{A'}{}^{B'}$	Normal connection
$T^{A'}$	Normal torsion
$R_{A'}{}^{B'}$	Normal curvature tensor
d'	Normal exterior derivative
$\mathcal{P}'$	Normal canonical one-form
∇'	Normal covariant derivative
h	World-sheet metric
e^A	World-sheet vielbein
$\omega_A{}^B$	World-sheet connection
$\mathscr{T}^A$	World-sheet torsion
$\mathscr{R}_A{}^B$	World-sheet curvature tensor
d	World-sheet exterior derivative
θ	World-sheet canonical one-form
$\mathscr{D}$	World-sheet covariant derivative

${K_A}^{B'}$	Extrinsic curvature
${L_A}^{B}$	Difference between world-sheet and intrinsic connection
$\tilde{\nabla}$	Diagonal connection
$\tilde{\theta}$	Inbetween worlds canonical one-form
$\mathscr{E}_A{}^{\underline{A}}$	Embedding matrix
$\mathscr{K}_A{}^{B'}$	Extrinsic curvature non o.n. base
$\mathscr{L}_A{}^{B}$	Difference between world-sheet and intrinsic connection non o.n. base
$\underline{\nabla}$	Derivative on both world-sheet and target space
$\underline{\mathscr{D}}$	World-sheet derivative on world-sheet vectors and target space derivative on target space vectors
$\underline{\hat{\nabla}}$	Modified covariant derivative
$\underline{\hat{\mathscr{D}}}$	Modified but world-sheet plus target space derivative
I	Almost product structure
$\underline{\tilde{\nabla}}$	Adapted connection
$\underline{\tilde{\tilde{\nabla}}}$	Vidal connection
$\underline{\tilde{R}}$	Adapted curvature tensor
$\underline{\tilde{\tilde{R}}}$	Vidal curvature tensor
$\underline{\tilde{T}}$	Adapted torsion tensor
$\underline{\tilde{\tilde{T}}}$	Vidal torsion tensor
H	Deformation tensor
H'	Complementary deformation tensor
L	Twisting tensor
L'	Complementary twisting tensor
K	Extrinsic curvature tensor
K'	Complementary extrinsic curvature tensor
κ	Mean curvature tensor
κ'	Complemenatry mean curvature tensor
W	Conformation tensor
W'	Complementary conformation tensor

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Paper I

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Monopole and Dyon Spectra in N=2 SYM with Higher Rank Gauge Groups

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Abstract: We derive parts of the monopole and dyon spectra for $N=2$ super-Yang–Mills theories in four dimensions with gauge groups G of rank $r \geq 2$ and matter multiplets. Special emphasis is put on $G=SU(3)$ and those matter contents that yield perturbatively finite theories. There is no direct interpretation of the soliton spectra in terms of naïve selfduality under strong–weak coupling and exchange of electric and magnetic charges. We argue that, in general, the standard procedure of finding the dyon spectrum will not give results that support a conventional selfduality hypothesis — the $SU(2)$ theory with four fundamental hypermultiplets seems to be an exception. Possible interpretations of the results are discussed.

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1. Introduction

The last years have seen a tremendous progress in the understanding of nonperturbative aspects of four-dimensional field theory. New techniques [1,2,3,4,5] enable calculation of exact results valid beyond the perturbative level. It was long ago conjectured [6,7] that the $N = 4$ supersymmetric Yang–Mills (SYM) theories should possess some kind of strong–weak coupling duality. These theories are perturbatively finite [8,9,10,11,12,13], and actually exactly finite [14]. Actual calculations of dyon spectra in these theories [15,16, 17,18], and also other tests [19] give strong support for the duality hypothesis. There are also an infinite number of theories, possessing $N=2$, but not $N=4$, supersymmetry that are one-loop, and thus perturbatively, finite. The only one of these theories that has undergone closer examination with respect to duality properties is the $SU(2)$ model with four hypermultiplets in the fundamental representation. There, all results confirm duality, and it is tempting to conclude that the same is true for all perturbatively finite $N=2$ SYM theories. Since all explicit calculations of BPS states in $N=4$ theories and the finite $N=2$ $SU(2)$ theory sofar are in excellent agreement with predictions from duality, it is natural to continue this program and include also the other perturbatively finite $N=2$ theories. The aim of this paper is to do this by calculating part of the dyon spectra for such theories. As we will demonstrate, a number of problems arise. They are partly associated with the lattice structures of electric and magnetic charges, and also with the inaccessibility of monopole–anti-monopole configurations.

In sections 2 and 3, basic properties about monopoles and their moduli spaces are reviewed. Section 4 applies an index theorem to find the dimensions of bundles of zero-modes of the various fields in the theories over moduli space. Section 5 contains a discussion on the lattice properties of electric and magnetic charges, giving a general argument against naïve duality. In section 6, the effective action for the monopoles is derived from the field theory, and some aspects of its quantization are discussed. Section 7 applies this quantization to some specific examples, and derives the corresponding dyon spectra. They do not support naïve duality. In section 8, the implications of the results are discussed.

2. Monopoles — Symmetry Breaking and Topology

In this section, we will give a quick review of the concept of Bogomolnyi–Prasad–Sommerfield (BPS) monopoles [20,21] and their topological properties, aiming at a topological description suited for the index calculations of section 4. A BPS (multi-)monopole is a static configuration of the Yang–Mills–Higgs (YMH) system that due to its topological character has a relation between mass (energy) and magnetic charge. Consider the hamiltonian of the YMH system with gauge group G (the Higgs field is in the adjoint representation):

$$H = \frac{1}{2}\int d^3x\, Tr\,(B_iB_i + D_i\Phi D_i\Phi) = \frac{1}{4}\int d^3x\, Tr\Big\{(B_i + D_i\Phi)^2 + (B_i - D_i\Phi)^2\Big\}\ . \qquad (2.1)$$

If the Bogomolnyi equation

$$B_i = \pm D_i\Phi \qquad (2.2)$$

is imposed (note that this equation alone implies that the equations of motion are satisfied), the energy becomes topological:

$$H = \pm \int d^3x \, Tr B_i D_i \Phi = \int_{\mathbb{R}^3} Tr F D\Phi = \int_{\mathbb{R}^3} Tr D(F\Phi) = \int_{S^2_\infty} Tr F\Phi \ , \tag{2.3}$$

and can be related to the topological magnetic charges of the field configuration (see below).

The topological information of the BPS configuration resides entirely in the asymptotic behaviour of the Higgs field. Let us denote the Higgs field at the two-sphere S^2_∞ at spatial infinity by $\phi(x)$. By a gauge transformation, it can always (locally) be brought to an element in the Cartan subalgebra (CSA) of $\mathfrak{g}$, the Lie algebra of G, and furthermore, by Weyl reflections, into a fundamental Weyl chamber. The equations of motion then imply that this element is constant on S^2_∞. We thus have

$$\psi = g^{-1}(x)\phi(x)g(x) \ , \tag{2.4}$$

where ψ is a constant element in the CSA. The group element $g(x)$ is not globally defined on S^2_∞, though ϕ and ψ are. If g is defined patchwise on the two hemispheres, the difference on the equator is an element in $H \subset G$, the stability group of ϕ. We will only consider the generic case of maximal symmetry breaking, when H is the maximal torus of G. This occurs as long as the diagonalized Higgs field ψ does not happen to be orthogonal to any of the roots. $H=(U(1))^r$ is the unbroken gauge group, where r is the rank of G. In the light of equation (2.4), the Higgs field on S^2_∞ may be viewed as a map from S^2_∞ to the homogeneous space G/H, and all the topological information now lies in the gauge transformation $g(x)$. The relevant classification is $\pi_2(G/H)$, which (for semisimple G) is isomorphic to $\pi_1(H)$. For the case at hand, this group is $\mathbb{Z}^r$, *i.e.* there are r magnetic charges. It is straightforward to calculate the vector k of magnetic charges. The gauge transformation (2.4) induces a connection $\omega = g^{-1}dg$ with field strength $f = d\omega + \omega^2 = 0$ locally but not globally (with the two patches defined above, f has distributional support on the equator), the magnetic charges of which can be expressed as

$$k\cdot T = \frac{1}{2\pi i}\int_{S^2} f = \frac{1}{2\pi i}\int_{S^1} (\omega_{\text{north}} - \omega_{\text{south}}) \tag{2.5}$$

(the last integral is evaluated at the equator of S^2 where the two patches of the connection meet). The mass of the configuration is expressed in terms of k using equation (2.3):

$$m = \pm \int_{S^2} Tr f\psi = 2\pi\,|h\cdot k| \tag{2.6}$$

where ψ is expressed in terms of the vector h as $\psi = h\cdot T \in$ CSA. In section 4, we will use the gauge transformation g in order to calculate indices of Dirac operators in a monopole background, yielding the number of zero-modes of certain fields in the presence of a monopole.

The magnetic charge vector obtained from equation (2.5) lies on the coroot lattice $\Lambda_r^\vee$ of G. This agrees with the generalized Dirac quantization condition on electric and magnetic charges, that

$$e\cdot k \in \mathbb{Z} \tag{2.7}$$

for any charge vectors e and k. Since e must lie on the weight lattice Λ_w of G, k must lie on the dual lattice of the weight lattice, *i.e.* the coroot lattice. We should comment on our choice of normalization for the magnetic charges. It means that the scale of the coroot lattice is chosen so that the coroots are

$$\Lambda_r^\vee \ni \alpha^\vee = \frac{2\alpha}{|\alpha|^2} \ , \tag{2.8}$$

and coincide with the roots for simply laced groups.

An elegant and convenient way of treating the YMH system in a unified way is to consider the Higgs field as the fourth component of a euclidean four-dimensional gauge connection. We thus let $A_4=\Phi$, and demand that no fields depend on x^4. It is useful to go to a quaternionic formalism, where the gauge connection sits in a quaternion $A=A_\mu e_\mu \in \mathbb{H}$, $e_4=1$ being the quaternionic unit element and e_i, $i=1,2,3$ the imaginary unit quaternions: $e_i e_j = -\delta_{ij}+\varepsilon_{ijk}e_k$. The Bogomolnyi equation (2.2) now becomes an (anti-)selfduality equation for the field strength $F_{\mu\nu}$:

$$F_{\mu\nu} = \pm\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{2.9}$$

and the topological character of the solutions becomes even more obvious. We will use the fact that a selfdual antisymmetric tensor can be expressed as an imaginary quaternion, and is formed from two vectors as $f^+=Im\,(vw^*)$. An anti-selfdual tensor is formed as $f^-=Im\,(v^*w)$. Spinors of both chiralities come as quaternions. The Weyl equations are for the s chirality $D^*s=0$ and for the c chirality $Dc=0$. For a more detailed discussion of the quaternionic formalism, transformation properties etc., see *e.g.* reference [22].

3. Moduli Spaces and Zero-Modes

A monopole solution is not an isolated phenomenon. There are always deformations of the field configuration that do not modify the energy. These always continue to satisfy the Bogomolnyi equation (2.2, 2.9). Deformations of the YMH system alone define tangent directions in the moduli space of monopole solutions at given magnetic charge k. One obvious set of such deformations is given by simply translating the (localized) solution. Therefore, the moduli space always contains a factor $\mathbb{R}^3$, but there are in general more possible moduli. Also, when other fields are present, as in the $N=2$ models we consider, these may also possess zero-modes in the BPS monopole background. These zero-modes also have to be considered in the low energy treatment we will make.

We will first give a resumé of some of the geometric aspects of the geometry of the moduli spaces (following reference [23], but in the quaternionic formalism of [22]), and then move on to the full $N=2$ model.

Suppose we search for a deformation δA of the gauge connection (in a quaternionic form, containing the Higgs field). The linearized version of the Bogomolnyi equation (with the plus sign — the anti-selfdual case is analogous) is $Im\,(D^*\delta A)=0$, where the rule for formation of an anti-selfdual tensor from from two vectors has been used. Denote the tangent directions by an index m. The natural metric is induced from the kinetic term in the action,

$$g_{mn} = \int d^3x\ Tr(\delta_m A_\mu \delta_n A_\mu) = \int d^3x\ TrRe\,(\delta_m A^* \delta_n A) \equiv \langle\delta_m A, \delta_n A\rangle \quad . \tag{3.1}$$

We would like a (physical) tangent vector to be orthogonal to any gauge modes in this metric, and therefore impose the supplementary condition $Re\,(D^*\delta A)=0$. The two conditions so derived for the deformations $\delta_m A$ are collected in

$$D^*\delta_m A = 0\ . \tag{3.2}$$

We note that this equation is formally identical to a Weyl equation for one of the four-dimensional spinor chiralities. It is also straightforward to show that the Weyl equation for the other chirality never can have L^2 solutions, simply because the background field strength is selfdual. The dimension of a moduli space at given k can therefore be calculated as the L^2 index of the Dirac operator on $\mathbb{R}^3$ in a known BPS background. As we will see, the only essential information that goes into the index calculation is the asymptotic behaviour of the Higgs field. This calculation will yield the complex dimension of the moduli space, *provided some selfdual solution with this asymptotic behaviour exists.*

All moduli spaces are known to be hyperKähler. The action of the complex structures on the tangent vectors is easily understood. If a tangent vector $\delta_m A$ satisfies equation (3.2), then also $\delta_m A e_i$ satisfy the same equation. The three complex structures act as

$$J^{(i)}{}_m{}^n \delta_n A = \delta_m A e_i\ . \tag{3.3}$$

They can be shown to be covariantly constant with respect to the connection derived from (3.1).

A parallel transport in the tangent directions of moduli space on the space of zero-modes should preserve the condition that tangent vectors are orthogonal to gauge modes. In order to achieve this, one introduces the gauge parameters $\varepsilon_m(x)$ and writes

$$\delta_m A = \partial_m A - D\varepsilon_m\ . \tag{3.4}$$

Parallel transport is generated by the covariant derivative $s_m = \partial_m + \mathrm{ad}\,\varepsilon_m$ (more generally, ε_m acts in the appropriate representation of the gauge group), with the property $[s_m, D] = \delta_m A$. This implies that $D^*_{A+dt^m\delta_m A}(\varrho + dt^m s_m\varrho) = 0$ for zero-modes in any representation, so that s_m provides a good parallel transport of all zero-modes. It is straightforward to calculate the Christoffel connection of the metric (3.1),

$$\Gamma^m{}_{np} = g^{mq}\int d^3x\ Tr\ \delta_q A_\mu s_n \delta_p A_\mu = g^{mq}\int d^3x\ Tr\,Re\,(\delta_q A^* s_n \delta_p A) = g^{mq} <\delta_q A, s_n\delta_p A>\ , \tag{3.5}$$

and the riemannian curvature [22],

$$\begin{aligned} R_{mnpq} &= <\delta_p A, [s_m, s_n]\delta_q A> \ +\ <s_m\delta_p A, \Pi_+ s_n \delta_q A> \ -\ <s_n\delta_p A, \Pi_+ s_m\delta_q A> \\ &= <\delta_p A, [s_m, s_n]\delta_q A> \ -4P_{+pq}{}^{rs} <\delta_m A, [s_n, s_r]\delta_s A>\ , \end{aligned} \tag{3.6}$$

where $\Pi_+ = D(D^*D)^{-1}D^*$ is the projection operator on higher modes and $P_{+pq}{}^{rs} = \frac{1}{4}J^{(a)}{}_{[p}{}^r J^{(a)}{}_{q]}{}^s$ is the projection operator on the part of an antisymmetric tensor that commutes with the complex structures, i.e. the $Sp(n)$ part, $4n$ being the real dimension ($J^{(4)}$ is defined as the unit matrix). The curvature is a $(1,1)$-form with respect to all three complex structures, which is equivalent to $Sp(n)$ holonomy, *i.e.* "selfduality".

The action for our $N=2$ super-Yang–Mills theory with matter is most conveniently formulated as the dimensional reduction of an $N=1$ theory in $D=(1,5)$. The six-dimensional action reads:

$$
\begin{aligned}
\mathscr{L} = -\frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}Re\,(\lambda^\dagger\Sigma^M D_M\lambda) \\
-\frac{1}{2}D_M q_f^* D^M q_f + \frac{1}{2}Re\,(\psi_f^\dagger\tilde{\Sigma}^M D_M\psi_f) + Re\,(\psi_f^\dagger\lambda q_f^*) + \frac{1}{8}(q_f^*\times q_f)^2\ .
\end{aligned}
\tag{3.7}
$$

Here, representation indices and traces have been suppressed for clarity. In addition to the gauge potential and its superpartner λ in the adjoint representation, there are the matter bosons q and fermions ψ. The subscript f labels the matter multiplets. A dagger denotes quaternionic conjugation and transposition, and, if the representations of G are complex, also complex conjugation. The matrices Σ and $\tilde{\Sigma}$ are six-dimensional quaternionic sigma matrices, and the cross product in the last term denotes Clebsh–Gordan coefficients for formation of an element in the adjoint representation. The fermions λ and ψ are two-component quaternionic spinors of opposite six-dimensional chiralities, and the matter boson q is a scalar quaternion.

The supersymmetry transformations are:

$$
\begin{aligned}
&\delta A_M = Re\,(\varepsilon^\dagger\Sigma_M\lambda)\ , && \delta q_f = \psi_f^\dagger\varepsilon\ , \\
&\delta\lambda = -\tfrac{1}{2}F_{MN}\tilde{\Sigma}^{MN}\varepsilon + \tfrac{1}{2}\varepsilon(q_f^*\times q_f)\ , && \delta\psi_f = \Sigma^M\varepsilon D_M q_f^*\ .
\end{aligned}
\tag{3.8}
$$

It is clear from the transformation of λ that a BPS background, obeying (2.2), breaks half the supersymmetry.

The Higgs field comes as one of the components (A_4, say) of the six-dimensional gauge connection. The euclidean four-dimensional formulation automatically comes out on reduction to four euclidean dimensions, upon which a spinor (of any six-dimensional chirality) splits into a pair of quaternionic spinors of opposite four-dimensional chiralities.

In order to examine which fields carry zero-modes in the BPS background, and go into a low energy expansion, we give the moduli parameters a slow time dependence and expand the equations of motion in the parameter $n = \#(\frac{d}{dt}) + \frac{1}{2}\#(\text{fermions})$. At $n=0$ one only has the background fields A with selfdual field strength. At $n=\frac{1}{2}$, there are the Weyl equations for the upper (s chirality) components of λ and ψ, which we denote α and β, respectively. Their lower (c chirality) components vanish to this order. The time dependence of the bosonic moduli is modeled so that $A=A(x,X(t))$. Then the equations at order $n=1$ imply, using $\dot{A}=\dot{X}^m(\delta_m A+D\varepsilon_m)$,

$$
\begin{aligned}
A_0 &= \dot{X}^m\varepsilon_m + (D^*D)^{-1}(-\alpha^*\alpha + \frac{1}{2}\beta_f^*\times\beta_f)\ , \\
A_5 &= (D^*D)^{-1}(\alpha^*\alpha + \frac{1}{2}\beta_f^*\times\beta_f)\ , \\
q_f^* &= -(D^*D)^{-1}(\alpha^*\beta_f)\ .
\end{aligned}
\tag{3.9}
$$

We see that the only fields that carry zero-modes, apart from the tangent directions to moduli space itself in the YMH system, are the fermions, both in the vector multiplet and the matter multiplets.

In order to get information about the number of fermionic zero-modes in the BPS background, we have to apply the index theorem of Callias [24] to the appropriate representations of λ (the adjoint) and ψ. We have already seen that the equation for tangent vectors to the moduli space is equivalent to a Weyl equation, so that the zero-modes of λ will come in the tangent bundle over moduli space, whose dimension is given by the index theorem. The zero-modes of ψ will come in some other index bundles with some connections. These connections and their curvatures are derived analogously to the riemannian curvature above. if the mode functions are denoted ϱ_α and normalized so that α is the fiber index of an orthonormal bundle, the connection is

$$\omega_{m\alpha\beta} = <\varrho_\alpha, s_m \varrho_\beta> \ , \tag{3.10}$$

and the curvature

$$F_{mn\alpha\beta} = <\varrho_a, [s_m, s_n]\varrho_\beta> + <s_m\varrho_\alpha, \Pi_+ s_n \varrho_\beta> - <s_n\varrho_\alpha, \Pi_+ s_m \varrho_\beta> \ . \tag{3.11}$$

4. Dimensions of Moduli Spaces and Index Bundles

Callias [24] has given an index theorem for the Dirac operator on $\mathbb{R}^{2n-1}$ in the presence of a gauge connection and a scalar matrix valued hermitean (Higgs) field that takes some nonzero values at spatial infinity. This index theorem is applicable precisely to the situation at hand. The index only depends on the (topological) behaviour of the Higgs field Φ at infinity. Callias theorem states that the L^2 index of the Dirac operator on $\mathbb{R}^3$ in the representation ϱ is given as

$$index \not{D}_\varrho = -\frac{1}{16\pi i}\int_{S^2_\infty} Tr_\varrho(UdUdU) \ . \tag{4.1}$$

Here, the matrix U is defined as $U = (\phi^2)^{-1/2}\phi$. Callias postulates that ϕ should have no zero eigenvalues, so that U is well defined. This assumption is directly related to the Dirac operator being Fredholm. If it does not have this property, there is a continuous spectrum around zero that, depending on the behaviour of the density of states, may contribute to the index calculation and give an incorrect result. Actually, in the case we are interested in, there are zero eigenvalues, corresponding to the fields that remain massless after the symmetry breaking. E. Weinberg [25] has shown that the massless vector bosons of the generic maximal symmetry breaking pattern do not contribute in the index calculation. On the other hand, for nonmaximal breaking to a nonabelian group H, one has to be more careful, and examine the exact contribution due to the roots orthogonal to the Higgs field. The same is true for some special values of the Higgs field that becomes orthogonal to some weight in a representation for the matter fields (see below). In the generic case, though, all one has to do is to replace the matrix ϕ by its restriction to the subspace spanned by the eigenvectors with nonzero eigenvalues. The corresponding restricted Dirac operator will have the desired Fredholm property.

The actual computation of the index is conveniently performed using the gauge transformation g of section 2. After the gauge transformation has been performed, the Higgs field has changed to the diagonalized Higgs field $\psi \in$ CSA, and the derivative simply becomes the commutator with the induced connection ω, since $d\psi = 0$. We thus have

$$index \not{D}_\varrho = -\frac{1}{16\pi i}\int_{S^2_\infty} Tr_\varrho(V[\omega, V]^2) = \frac{1}{4\pi i}\int_{S^2_\infty} Tr_\varrho(V d\omega) \ , \tag{4.2}$$

where $V=(\psi^2)^{-1/2}\psi$, and we have used $V^2=1$ and $d\omega=-\omega^2$. Taking the trace in the representation ϱ gives the result, using equation (2.5) for the magnetic charge vector,

$$\begin{aligned} &index \not{D}_\varrho = k\cdot\Lambda\ ,\\ &\Lambda = \frac{1}{2}\sum_{\lambda\in\varrho}\lambda\, sign(h\cdot\lambda)\ , \end{aligned} \tag{4.3}$$

the sum being performed over the weights of the representation ϱ. It is clear that the index stays constant as long as h does not become orthogonal to some weight, in which case the index changes discontinuously.

The expression (4.3) enables us to calculate the index explicitly for any magnetic charge and any representation of G. We will now turn to some examples that will be of use later. We first define the simple roots with respect to the value of the diagonalized Higgs field in the Cartan subalgebra. The vector h can always be chosen in the fundamental Weyl chamber so that its scalar product with all simple roots is positive. This is illustrated for $SU(3)$ in figure 1.

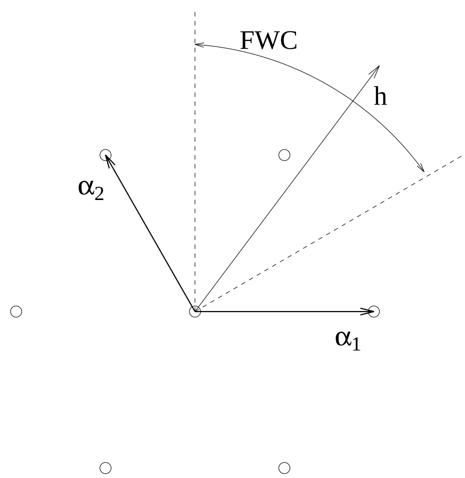

Figure 1. The fundamental Weyl chamber and the simple roots for $SU(3)$.

Starting with the adjoint representation, it may be verified that when the magnetic coroot vector is expressed as a linear combination of the simple coroots (with the normalization (2.8)) as

$$k = k_1\alpha_1^\vee + k_2\alpha_2^\vee + \ldots + k_r\alpha_r^\vee\ , \tag{4.4}$$

the index for the Dirac operator is

$$index \not{D}_{\mathrm{adj}} = 2\,(k_1 + k_2 + \ldots + k_r) \tag{4.5}$$

for any semisimple Lie group (and maximal symmetry breaking).

In order to translate this result into the complex dimension of a moduli space at magnetic charge k, some care has to be taken — it is only true provided that some selfdual configuration with the corresponding asymptotic behaviour of the Higgs field actually exists (or anti-selfdual, so that the dimension is minus the index). The result indicates that the real dimension of a moduli space for k a simple coroot is 4. This can be verified — such selfdual solutions exist, and are described by embeddings of the 't Hooft–Polyakov [26,27] $SO(3)$ monopole. We denote these simple monopoles. According to the interpretation of E. Weinberg [25], any multi-monopole at a k given by (4.4) with only positive coefficients k_i can in an asymptotic region be approximated by a superposition of well separated simple monopoles, and analogously for anti-monopoles. This agrees with the linearity of the index in k. A magnetic coroot formed as (4.4) with both positive and negative coefficients would asymptotically correspond to a field configuration that is approximately selfdual in some regions and anti-selfdual in others. Such a configuration can not be static, since the magnetic and Higgs forces between a monopole and an anti-monopole do not cancel. Either such configurations do not exist, or they are simply inaccessible to us at our present understanding. This is of course a problem already with gauge group $SU(2)$, but there it does not manifest itself in terms of allowed and disallowed sectors in the coroot lattice, as it does for higher rank gauge groups, merely as a lack of understanding of the interaction between monopoles and anti-monopoles. If one doubts the above argument, it is illuminating to consider the points in the coroot lattice where the index (4.5) vanishes. Since the dimensionality of a moduli space can not be zero (translations are always moduli), it becomes clear that no static BPS configurations with these magnetic charges can exist. The allowed sectors for magnetic charges in an $SU(3)$ theory are shown in figure 2, where unfilled roots indicate forbidden magnetic charges.

Another representation of special interest is the fundamental representation of $SU(N_c)$. Beginning with $SU(3)$, and ordering the (co)roots by $h\cdot\alpha_1 > h\cdot\alpha_2$, the index becomes $index\not{D}_{3(SU(3))} = k_1$. This is the complex dimension of the fiber of the index bundle of zero-modes in the fundamental representation for allowed positive magnetic charges. We note that when the Higgs field aligns with the root $\alpha_1+\alpha_2$ in the middle of the fundamental Weyl chamber, a quark and an antiquark become massless, and the index formula of Callias may give the wrong result. In fact, when h crosses this line, the zero-mode at $k=\alpha_1$ disappears and a new zero-mode instead appears at $k=\alpha_2$. The index formula gives a result in between, which clearly is nonsense. The Dirac operator is not Fredholm in the fundamental representation in this case. It is possible, though, to follow the asymptotic behaviour of the solutions to the Dirac equation. For generic h the normalizable solutions decay exponentially with the radius, while for a degenerate case as this one there is a power law behaviour. One may check that these solutions have a leading term proportional to $r^{-1/2}$, so they are not L^2. For this special direction of the Higgs field there are thus no zero-modes in the fundamental representation. A similar situation occurs at the boundary of the fundamental Weyl chamber, where the symmetry breaking pattern changes to $H=SU(2)\times U(1)$ as some vector bosons become massless. We do not consider this nonmaximal symmetry breaking in this paper.

The indices in the fundamental representations of other $SU(N_c)$ groups behave in a similar way. We can illustrate by looking at $SU(4)$, where we have the simple roots $\alpha_{1,2,3}$ with $\alpha_1^2=\alpha_2^2=\alpha_3^2=2$, $\alpha_1\cdot\alpha_2=\alpha_2\cdot\alpha_3=-1$, $\alpha_1\cdot\alpha_3=0$. In the interior of the fundamental Weyl chamber the symmetry breaking pattern is the maximal one, $SU(4)\to U(1)\times U(1)\times U(1)$. At the three planes forming the boundary, $SU(4)$ is broken to $SU(2)\times U(1)\times U(1)$, and where the planes intersect to $SU(2)\times SU(2)\times U(1)$ (one line) or $SU(3)\times U(1)$ (two lines). The weights in the representation 4, specified by their scalar products with the simple roots, are $\lambda_{(1,0,0)}$, $\lambda_{(-1,1,0)}$, $\lambda_{(0,-1,1)}$ and $\lambda_{(0,0,-1)}$. The fundamental Weyl chamber divides in two parts, related by the $\mathbb{Z}_2$ of outer automorphisms, and we choose to stay in the region where $h\cdot\alpha_1 > h\cdot\alpha_3$. On the boundary there are massless quarks. This also happens when $h\cdot\lambda_{(-1,1,0)}=0$. This plane divides the half fundamental Weyl chamber in two parts. In the region where $h\cdot\lambda_{(-1,1,0)}>0$ the index is $index\not{D}_{4(SU(4))}=k_2$ and when $h\cdot\lambda_{(-1,1,0)}<0$ it is $index\not{D}_{4(SU(4))}=k_1$. Similar statements hold for higher $SU(N_c)$ groups. The index in the fundamental representation depends only on one of the simple magnetic charges.

The representation 6 of $SU(3)$ is interesting because it is contained in one of the perturbatively finite models. The index is $index\not{D}_{6(SU(3))} = 3k_1$, with the same choice of ordering of the roots as above. The number of complex zero-modes for the representations 3, 6 and 8 of $SU(3)$ in the allowed sector of positive k are shown in figure 2 (negative k are obtained by reflection in the origin).

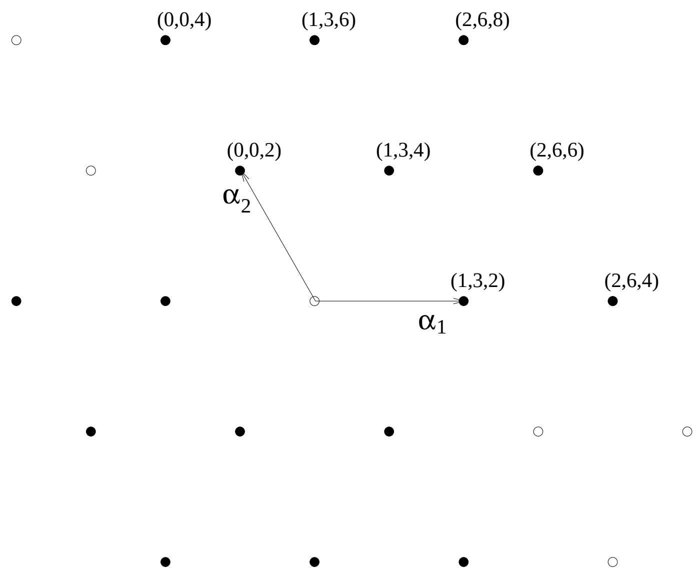

Figure 2. The allowed positive magnetic charges and number of zero-modes in 3, 6 and 8 of $SU(3)$.

5. Lattices of Electric and Magnetic Charges — Duality?

In this section, we will comment on the possibilities for dual theories from the viewpoint of electric and magnetic charge lattices. In the original Goddard–Nuyts–Olive (GNO) duality conjecture [7], which generalizes the Montonen–Olive conjecture [6] of $SO(3)$ to higher rank gauge groups (both applying to $N=4$ SYM), it is noted that the magnetic charges lie on the coroot lattice $\Lambda_r^\vee$, which is the root lattice of the "dual group", *i.e.* the group where long and short roots are interchanged. It should be noted, for clarity, that even if the spectrum of electric charges of the elementary excitations of the theory does not span the entire weight lattice (of $SU(N_c)$, say), this does not leave us with more choices for the magnetic charges, as one might suspect from the generalized Dirac condition (2.7). Indeed, the magnetic charges still are constrained to the coroot lattice, as seen from equation (2.5), disregarding of the matter content of the theory. So, in the case of $N=4$ SYM with gauge group $SU(N_c)$, where all fields come in the adjoint representation, *i.e.* only in one of the N_c conjugacy classes of the weight lattice, so the actual gauge group is $SU(N_c)/\mathbb{Z}_{N_c}$, the magnetic charges still lie on the coroot lattice of $SU(N_c)$, *i.e.* on the weight lattice of $SU(N_c)/\mathbb{Z}_{N_c}$. The GNO conjecture states that an $N=4$ SYM theory is $\mathbb{Z}_2$ dual to the $N=4$ SYM theory with the dual gauge group. The validity of the conjecture has been partially vindicated by actual calculation of parts of the dyon spectra [15,16,17,18].

When we consider $N=2$ models with a matter content that makes the theory perturbatively finite, the GNO interpretation of the coroot lattice must be revised. For example in the $SU(2)$ theory with four fundamental hypermultiplets, the coroot lattice of $SU(2)$ is reinterpreted as the weight lattice of $SU(2)$ instead of the root lattice. This simply amounts to a rescaling by a factor 2. The $\mathbb{Z}_2$ pictures of the quarks now reside at $k=\pm\alpha^\vee$, where in the $N=4$ theory the duals of the massive vector bosons were found. This is of course possible due to the simple fact that the root and weight lattices of $SU(2)$ are isomorphic up to an overall scale. Some of the dyonic states with low magnetic charges have been found, and support the duality hypothesis [2,28,29].

When we move to more general gauge groups, the picture is less clear. As a first example, we have the two perturbatively finite $SU(3)$ theories, one with six hypermultiplets in the fundamental representation, the other with one fundamental multiplet and one in the representation 6. The elementary excitations now carry electric charges in all three conjugacy classes of $SU(3)$, so we want also the magnetic charges to fill out the entire weight lattice of $SU(3)$, if $\mathbb{Z}_2$ duality is supposed to hold. This reinterpretation of the coroot lattice is indeed possible, since the root and weight lattices of $SU(3)$ are isomorphic up to a scale. It is therefore meaningful to examine the actual spectrum of monopoles and dyons in these two models in order to find signs for or against strong–weak coupling duality. As we will see later, the dyon spectra do not support naïve selfduality.

In general, already considering the lattices seems to contradict naïve duality. The coroot lattice, being the root lattice of the dual group, is generically not isomorphic to a weight lattice containing representations that allow matter multiplets in other conjugacy classes than the trivial one. Take $SU(4)$ as an example. The (co)root lattice of magnetic charges is the fcc lattice, while the weight lattice (dual to the (co)root lattice) of electric charges is the bcc lattice. With the GNO interpretation of the coroot lattice, the dual gauge group is $SU(4)/\mathbb{Z}_4$ and there is no room for matter in nontrivial conjugacy classes. The only possible matter content is in the adjoint representation, yielding the $N=4$ theory. One might look for a dyon spectrum that only contains states on some sublattice of the coroot lattice, isomorphic to the relevant part of the weight lattice [30]. Such sublattices exist, but as we will show explicitly (with $SU(4)$ as an example), the dyon spectrum is not confined to such a sublattice. Again, we recognize no signs of selfduality in the dyon spectrum.

Another point, already touched upon in section 4, is that even if the isomorphism between the root and weight lattices for $SU(3)$ is used as above, or if one tries to pick out a sublattice isomorphic to (part of) the weight lattice, one is immediately led to considering states in "forbidden sectors", asymptotically consisting of superpositions of monopoles and anti-monopoles. Such configurations are not included in the present treatment. Whether this is a fundamental impossibility or an incompleteness of the semi-classical procedure is not clear to us (there might exist non-static configurations that are possible to interpret as bound states of monopoles and anti-monopoles, although it is unclear to us how such states could saturate a Bogomolnyi bound).

6. The Effective Action — Quantization

The procedure we follow in order to find the soliton spectrum of the full quantum field theory is to make a low energy approximation of the theory in a BPS background. Due to the mass gap, corresponding to the Fredholm property of section 4, the number of degrees of freedom in this approximation becomes finite. The field configuration moves adiabatically on the moduli space, and the behaviour of the model is that of a supersymmetric quantum mechanical model with the moduli space as target space. The number of supersymmetries is half the number of supersymmetries in the original field theory. The reason why this low energy approximation gives reasonable information about the spectrum of the full theory is that if we find BPS saturated states at low energy, these will come in a short multiplet of the $N=2$ supersymmetry algebra, and will necessarily continue to do so at any scale [31]. If the theory is finite, the mass formula of the adiabatic approximation will be exact, while, if the theory is renormalizable, it is renormalized.

In order to find the supersymmetric quantum mechanical model corresponding to the actual theory we are interested in, we only keep the zero-modes of sections 3 and 4 as dynamical variables. Concretely, we derive the low energy action by solving for all fields to order $n=1$ as in (3.9), plug the solutions back into the field theory action (3.7), and keep terms of order $n=2$. We then integrate over three-space, using the expressions for metrics, connections and curvatures of section 3. The resulting lagrangian was calculated in [22], and reads

$$\mathscr{L} = -2\pi h\cdot k + \frac{1}{2} g_{mn}\dot{X}^m\dot{X}^n + \frac{1}{2} g_{mn}\lambda^m D_t\lambda^n + \frac{1}{2}\psi^\alpha D_t\psi^\alpha - \frac{1}{4}F_{mn\alpha\beta}\lambda^m\lambda^n\psi^\alpha\psi^\beta \ , \qquad (6.1)$$

Here, we have denoted the fermionic variables, in sections of appropriate bundles over moduli space, with the same letters that were used in the field theory action. The covariant derivatives used on the fermions are defined as $D_t\lambda^m = \dot{\lambda}^m + \Gamma^m{}_{np}\dot{X}^n\lambda^p$ and $D_t\psi^\alpha = \dot{\psi}^\alpha + \omega_m^{\alpha\beta}\dot{X}^m\psi^\beta$. If one has $N=4$ supersymmetry, also the ψ's come in the tangent bundle, and the field strength F is the riemannian curvature. The lagrangian (6.1) has "$N=\frac{1}{2}\times 4$" supersymmetry, meaning that there are four real supersymmetry generators. They take the form

$$Q(a) = \lambda^m J^{(a)}{}_m{}^n V_n \ , \qquad (6.2)$$

where V_m is the velocity $g_{mn}\dot{X}^n$. It is essential, and a necessary consequence of the existence of these supersymmetries, that F is selfdual, *i.e.* a $(1,1)$-form with respect to all three complex structures.

When quantizing the supersymmetric quantum mechanical system given by (6.1), we look for "ground states", *i.e.* states that continue to saturate a Bogomolnyi bound. These are zero energy states for the system given by the lagrangian without the first term, at least when the electric charge vanishes. The electric charges modify equation (2.6) to

$$m^2 = (h\cdot e)^2 + (\frac{2\pi}{g^2}\, h\cdot k)^2 \ , \qquad (6.3)$$

where the coupling constant has been reinstated explicitly (this relation follows from the form of the extension of the $N=2$ supersymmetry algebra). Consider the solutions (3.9) to the field equations. We can use them to derive an explicit expression for the electric charge density:

$$D_iE_i = \dot{X}^m D_i\delta_m A_i + \alpha^*\alpha - \frac{1}{2}\beta^*\times\beta \ . \qquad (6.4)$$

Integrating this over three-space gives a "topological" electric charge from the first term, which is the momentum on the S^1 of the moduli space. Here, the contribution to the charge density is $\dot{X}^4 D_iD_i\Phi$, and the electric field is proportional to the magnetic field, with the proportionality constant being the velocity on S^1, so that electric charges that arise this way are collinear with the magnetic ones. The second term does not contribute. Using $\alpha=\delta_m A\lambda^m$, where λ^m are real fermionic oscillators, it gives after integration $\lambda^m\lambda^n <\delta_m A^*\delta_n A> = 0$. Using $\beta=\varrho_\alpha\psi^\alpha$, the last term becomes $\psi^\alpha\psi^\beta <\varrho^*_\alpha\times\varrho_\beta>$. For a complex representation, this may contain an element in the

Cartan subalgebra. A straightforward calculation, using the orthogonality relations for the zero-modes of the fundamental representation of $SU(3)$ and magnetic charge α_1, shows that it indeed is $Q\bar{\psi}\psi$, where Q is the U(1) charge of the representation 2 in the decomposition $3 \to 2_{1/6} \oplus 1_{-1/3}$ under $SU(3) \to SU(2)\times U(1)$, the SU(2) being defined by α_1 as in the following section.

A comment on the mass–charge relation: When we find a quantum mechanical state using the low energy action, we can not expect to find the exact expression for the mass from the corresponding hamiltonian. What we see is a low energy approximation. For the S^1 momenta, it gives the first term in the series expansion for low velocity on the circle. For the "orthogonal" charges from the matter fermions, there is no continuous classical analogue, and these electric charges are not seen in the low energy hamiltonian. However, we can deduce from the fact that the states come in short multiplets that they must be BPS saturated.

One has to divide the fermionic variables into creation and annihilation operators. Using the Kähler property, we can take $\bar{\lambda}^{\bar{\mu}}$ as creation operators and λ^{μ} as annihilation operators, where μ is a complex index. We then have two equivalent pictures: either the states are forms with anti-holomorphic indices, or we view λ^m as gamma matrices as in the quantization of the spinning string, and the states are Dirac spinors. The equivalence is easily seen from a representation point of view — when the full holonomy $SO(4n)$ is reduced to $SU(2n)$, the two spinor chiralities decompose into even and odd forms. Zero energy states are harmonic forms, or spinors satisfying the Dirac equation.

The presence of the ψ's means that the forms/spinors have to be harmonic with respect to the connection ω, and also carry antisymmetric indices coming from the creation operator part of ψ (or a spinor index). In the case of $N=4$, the fermions together come in the complexified tangent bundle, so that ground states are any harmonic forms.

The general pattern is that the part of the λ's belonging to the $\mathbb{R}^3\times S^1$ part of moduli space generates the appropriate number of states of a short multiplet of the space-time supersymmetry algebra. We thus only have to consider the internal space (and only count singlets under the discrete group that is divided out) in order to find the number of multiplets. When the dimension of the internal space is four, one can use the selfduality of the field strength for a vanishing theorem, completely analogous to the one used in space-time: all the solutions to the Dirac equation have to come in only one of the spinor chiralities. This reduces the problem of identifying the ground states to that of calculating the index of the Dirac operator. For higher-dimensional moduli spaces, there is a priori no such vanishing theorem, and it seems like one has to resort to calculating the L^2 cohomology, which of course is a much harder problem, of which little seems to be known.

7. Dyon Spectra for Low Magnetic Charges

The moduli spaces for the magnetic charge being any simple coroot is identical to the one-monopole moduli space in the $SU(2)$ theory. Also, when k is a multiple of a simple coroot, the moduli space is identical to the corresponding $SU(2)$ moduli space. The new ingredient for higher rank gauge groups comes when k is a linear combination of different simple coroots. If k is a linear combination of orthogonal simple coroots, the moduli space factorizes metrically into the product of $SU(2)$ moduli spaces. The only nontrivial example that is accessible so far is the space at $k=\alpha^\vee+\beta^\vee$, where $\alpha^\vee\cdot\beta^\vee<0$. As is pointed out in [17,18], a very general argument tells us that the isometry group of the inner moduli space has to be $SU(2)\times U(1)$ (the "extra" $U(1)$ isometry is associated with local conservation of the "relative" magnetic charge). The unique regular hyperKähler manifold with this isometry is Taub–NUT with positive mass parameter, and global considerations (see Appendix A) lead to to the moduli space

$$\mathscr{M} = \mathbb{R}^3 \times \frac{S^1 \times \text{Taub–NUT}}{\mathbb{Z}_2} \ . \tag{7.1}$$

Appendix B contains some basic facts about Taub–NUT space.

We also would like to find explicit expressions for the connections and curvatures of the index bundles associated with the various matter fermions. Starting with the fundamental representation of $SU(3)$ as a model example, we consider the magnetic charge $k = \alpha_1$.

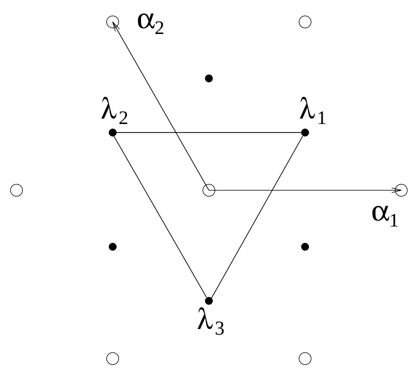

Figure 3. The representation 3 of $SU(3)$.

It is clarifying to calculate the index for the Dirac operator using a decomposition into $SU(2)\times U(1)$, where the $SU(2)$ is defined by the root α_1. The decomposition of the representation 3 is $3 \to 2_{1/6} \oplus 1_{-1/3}$. Only the 2 of $SU(2)$ (containing the weights λ_1 and λ_2 of figure 3) has a zero-mode, so that the zero-modes carry a $U(1)$ electric charge $1/6$. The S^1 in the moduli space is generated by gauge transformations with (the SU(2) part of) the Higgs field as gauge parameter. Already when this transformation arrives at the group element $\exp(\pi i \alpha_1 \cdot T)$, the nontrivial element in the center of $SU(2)$, it acts as the identity in the adjoint representation. In the fundamental representation of $SU(2)$, on the other hand, this element acts as minus the identity, which means that the index bundle has a $\mathbb{Z}_2$ twist around the S^1 [32]. This is true also here. If one imposes single-valuedness of the wave function, this implies that there is a correlation between the S^1 momentum, which is the electric charge in the α_1 direction and the excitation number of the ψ's, carrying electric charge in the direction orthogonal to α_1. The result of these considerations is that the electric charges are constrained to lie on the weight lattice, which is of course expected. The electric spectrum at $k = \alpha_1$ for the theory with six fundamental hypermultiplets is indicated in figure 4. The numbers denote representations under the flavour $SU(6)$.

The representation 6 is treated similarly. It decomposes as $6 \to 3_{-1/3} \oplus 2_{1/6} \oplus 1_{2/3}$. The 3 carries two zero-modes (in the tangent bundle) and the 2 one. This last zero-mode again has a $\mathbb{Z}_2$ twist around S^1. The electric spectrum at $k = \alpha_1$ for the model with one fundamental hypermultiplet and one in the representation 6 is depicted in figure 5, where the number of multiplets at each lattice point is indicated.

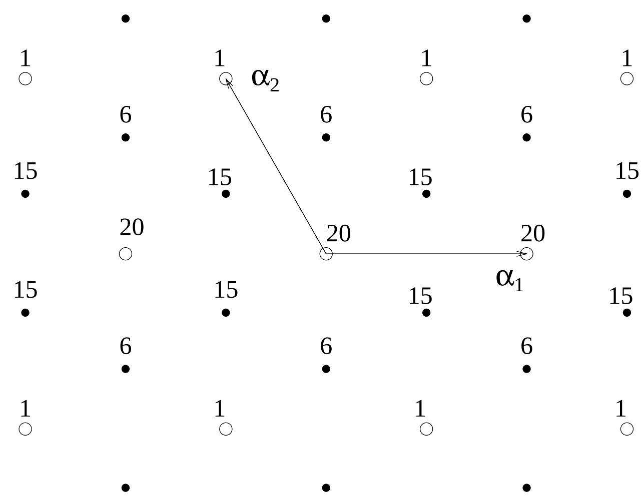

Figure 4. The electric spectrum at $k = \alpha_1$ for six multiplets in 3 of $SU(3)$.

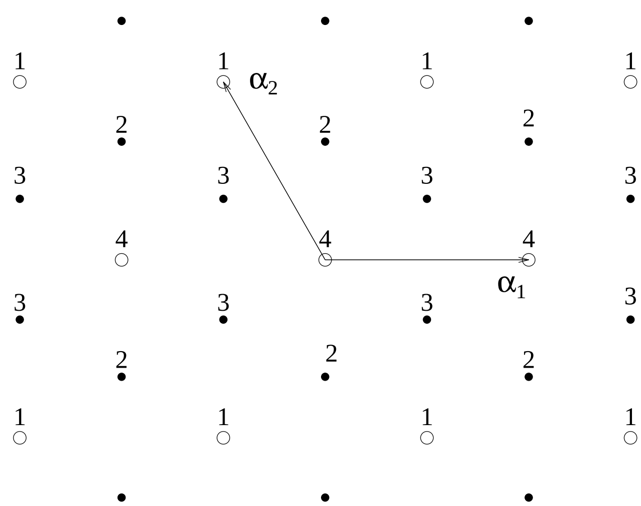

Figure 5. The electric spectrum at $k = \alpha_1$ for one multiplet in 3 and one in 6 of $SU(3)$.

At $k = \alpha_2$, there are no matter zero-modes. We just get one multiplet of states at electric charges that are multiples of α_2. The same statement holds true in the presence of matter in the representation 6.

At $k = \alpha_1 + \alpha_2$, there is one zero-mode in the fundamental representation. We have to find the connection of the index bundle over the Taub–NUT space. It has to be a $U(1)$ connection with selfdual field strength. It is a well known fact that there exists only one (linearly independent) selfdual harmonic two-form on Taub–NUT space, to which the field strength then has to be proportional (see appendix B). We have to determine the normalization factor c in front of the connection. This can be done by considering the holonomy in the region of moduli space where the monopoles are well separated, *i.e.* at large r. If we move around the circle generated by $\frac{\partial}{\partial\psi}$, the first time we should get back to the original configuration is after completing the whole circle.

Integrating along a curve $C_\gamma\,:\,0\leq\psi\leq\gamma$ at constant r gives $\int_{C_\gamma}\omega=\gamma c\frac{r-M}{r+M}$. Thus, the smallest value of γ for which $\exp(i\int_{C_\gamma}\omega)=1$ at infinite radius should be 4π. This gives $\oint_{C_{4\pi}}\omega=2\pi$, and $c=\frac{1}{2}$. We then need to find the index of the Dirac operator for fields of various charges with respect to the $U(1)$ connection. This is completely analogous to the calculation performed in [28,29] for the Atiyah–Hitchin manifold. One can use the Atiyah–Patodi–Singer index theorem and push the boundary to infinity. An additional issue here is that if we want to know the spectrum of the electric charge orthogonal to $\alpha_1+\alpha_2$, we must investigate how the solutions depend on the coordinate ψ. Luckily enough, both the index and the explicit expressions for the mode functions are known [33]. If we call the charge of one creation operator for the matter fermions 1, the states will come with charges q which are the "vacuum charge" plus n, where n is the number of creation operators applied. When the number of matter multiplets is even (we consider self-conjugate electric spectra) these charges will be integers. Pope [33] showed that the number of zero-modes of the Dirac operator for positive charge q is $\frac{1}{2}q(q+1)$ and that they depend on the ψ coordinate as $\exp(-\frac{1}{2}i\nu\psi)$, $\nu=1\ldots q$, the number of states at each value of ν being ν, together with an analogous statement for negative q. Taub–NUT space is simply connected, so the charges are a priori not restricted by any quantization rule, and the results in [33] contain this more general case. The value of ν is related to the electric charge in the direction orthogonal to $\alpha_1+\alpha_2$ by $Q=\nu/6$ with the normalization for the $U(1)$ charge used earlier. The $\mathbb{Z}_2$ identification of the moduli space produces a correlation between ν and the S^1 momentum. The spectrum of electric charges for $k=\alpha_1+\alpha_2$ in the $SU(3)$ model with six fundamental hypermultiplets is depicted in figure 6, where the numbers indicate $SU(6)$ representations.

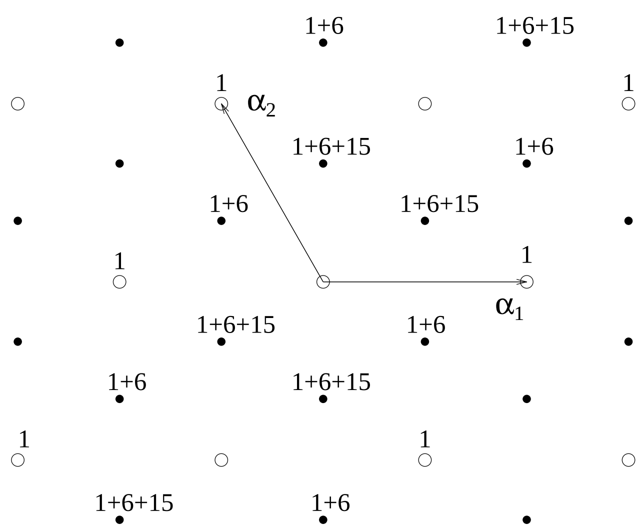

Figure 6. The electric spectrum at $k=\alpha_1+\alpha_2$ for six multiplets in 3 of $SU(3)$.

It is probably reasonably straightforward to derive the spectrum at $k=\alpha_1+\alpha_2$ also in the presence of matter in the representation 6. We have not done this.

The results for the $SU(3)$ theory with six fundamental hypermultiplets are summarized in figures 4 and 6, together with the electric spectrum at $k=\alpha_2$, which just consists of one multiplet at any integer multiple of α_2. It is also straightforward to extend the dyon spectrum to $k=2\alpha_1$

and $k=2\alpha_2$. We have not been able to find the states at the magnetic charges where $\mathbb{Z}_2$ duals of the vector bosons would be expected to reside. These are either outside of the allowed sectors or have eight-dimensional inner moduli spaces, whose metrics are not known. If we examine the electrically uncharged states at the magnetic charges where the dual quarks were expected, we find, instead of six multiplet at each lattice point, twenty, one and zero multiplets at charge α_1, α_2 and $\alpha_1+\alpha_2$ respectively.

For theories with $N=4$, the calculations are simpler. As shown in section 6, there are no fermion contributions to the electric charges, so the electric charge aligns with the magnetic charge. As already mentioned, ground states correspond to any (normalizable) harmonic forms on the internal moduli space. For simple magnetic coroots, the moduli space is $\mathbb{R}^3\times S^1$, and there is only one short multiplet for electric charges at integer multiples of the corresponding root (note that the integer in Dirac's quantization condition (2.7) is even). For k at twice a coroot, there is, as demonstrated by Sen [15], a unique selfdual harmonic two-form on the Atiyah–Hitchin manifold, corresponding to one multiplet at e being any odd multiple of the root (the selection of odd multiples comes from the $\mathbb{Z}_2$ divided out in the definition of the moduli space). At k being the sum of two simple coroots with negative scalar product, one has, as noted in [17,18], again the unique selfdual harmonic two-form mentioned earlier, that now gives one multiplet at any integer multiple of the corresponding root. Porrati [16] has presented convincing evidence for the existence of *all* states predicted by $Sl(2;\mathbb{Z})$ duality for the $N=4$ $SU(2)$ model. Note that the $Sl(2;\mathbb{Z})$ duals of the massive vector bosons in any $N=4$ theory always lie in the allowed sectors for the magnetic charges.

When we continue this discussion to higher rank gauge groups, nothing changes in principle. Part of the above discussion applies to moduli spaces at simple coroots or sums of two simple coroots for any gauge group. We have also seen (for the $SU(N_c)$ groups) that the matter in the fundamental representation behaves very similarly to what it does in $SU(3)$. This means that we can not hope to find dyon spectra with magnetic charges confined to a sublattice isomorphic to the weight lattice. There will always be some states at the simple coroots, which will not be in such a sublattice.

8. Conclusions and Outlook

The results of this paper are essentially the following. In spite of the success of the procedure applied here in finding the (low lying) dyon states predicted by $Sl(2;\mathbb{Z})$ duality for the $N=4$ models and the $N=2$ $SU(2)$ model with four fundamental hypermultiplets, the picture we see for higher rank gauge groups and matter content making the theory perturbatively finite is much less clear. We have for example not been able to identify the purely magnetically charged states in the quantum theory with the elementary excitations of some "dual" finite $N=2$ theory. There are also sectors of the magnetic charge lattices that are inaccessible due to our inability of treating systems containing monopoles and anti-monopoles, and this seems to exclude the treatment of states needed for duality. This is no problem for the $N=4$ theories, since the states needed for duality align with the roots, and are always found in the allowed sectors, but renders the situation problematic for $N=2$ models with gauge groups of rank $r\geq 2$.

As we see it, there are a couple of possible interpretations of the results of this paper. One is that the procedure in some way is incomplete. We saw that some of the magnetic charges we would need for a duality conjecture lie in forbidden sectors, that would correspond to superpositions of monopoles and anti-monopoles, something that is not accessible even in the $SU(2)$ models. We do not know how to describe scattering of monopoles and anti-monopoles, unless we move to a dual picture. On the other hand, if such configurations were relevant, they would enter at any magnetic charge, and they would probably modify the successful calculations supporting duality for the finite $SU(2)$ model. We find it unlikely that this could explain any shortcomings. In addition, we have seen that the lattice structures have problems that such a modification hardly could overcome.

A very drastic explanation of the results would be that the theories under consideration are not finite — that there would be instanton corrections to the β function, although one has perturbative finiteness. This sounds very strange and quite unlikely to us, but to our knowledge instanton contributions have not been calculated. On the other hand, the methods of [1, 2] have been applied to the case of $SU(N_c)$ with fundamental matter [34,35,36,37], and these results, support exact finiteness (although some of the statements are conflicting). It should be possible to perform at least a one-instanton calculation in order to verify that these models also are nonperturbatively finite.

A last possibility, which seems most likely, is that there is some kind of modified version of duality that does not include the $\mathbb{Z}_2$ of strong–weak coupling. A consideration that might give a clue is the following. The duality group has been conjectured to be not only $Sl(2;\mathbb{Z})$, but $Sp(r;\mathbb{Z})$, where r is the rank of the gauge group. When we examine the dyon spectrum of the $N=4$ theories, on the other hand, we only find electric charge vectors aligned with the magnetic ones (this is a direct consequence of the properties of monopole configurations at a multiple of a coroot, being embedded $SU(2)$ monopoles), so that we see only $Sl(2;\mathbb{Z})$ pictures of the elementary excitations. When we move to $N=2$ theories with higher rank groups, the "off-diagonal" part of the $Sp(r;\mathbb{Z})$ matrices, *i.e.* the one exchanging electric and magnetic charge, consists of a tensor in $\Lambda_r^\vee \otimes \Lambda_r^\vee$ and one in $\Lambda_w \otimes \Lambda_w$. Of course, in a suitable basis, these just become matrices with integer entries, but when the basis vectors for the two lattices are not aligned (which they in general are not, since the lattices are different) such a basis is not natural, in view of the mass formula (6.3). This means that in general, and even for $SU(3)$, there is no "natural" way of chosing an $Sl(2;\mathbb{Z})$ subgroup of $Sp(r;\mathbb{Z})$, where the tensors mentioned above would become diagonal. A supposed $\mathbb{Z}_2$ duality would in turn lie in such an $Sl(2;\mathbb{Z})$ subgroup. One might then speculate in some kind of "duality" for higher rank gauge groups that actually does not include a $\mathbb{Z}_2$ of electric–magnetic exchange. We find this issue interesting to pursue. In connection it is also worth mentioning that peculiar lattice properties of the charges in higher rank gauge groups have been found earlier. In reference [38], the existence of simultaneously massless dyons with nonvanishing $Sp(r;\mathbb{Z})$ product was demonstrated (for gauge group $SU(3)$), so that there should exist vacua where elementary excitations couple both electrically and magnetically to the gauge field. The evidence points towards a quite rich and interesting structure of these theories.

In conclusion, the results of this paper, rather than giving definite answers, raises a number of questions we find it urgent to investigate.

Note added: After correspondence with the authors of reference [37], we realize that for gauge groups of rank 2, and only then, there is a "natural" $\mathbb{Z}_2$ transformation, namely where the above mentioned tensors are the "epsilon tensors" $\alpha_1^\vee \otimes \alpha_2^\vee - \alpha_2^\vee \otimes \alpha_1^\vee$ and $\lambda_1 \otimes \lambda_2 - \lambda_2 \otimes \lambda_1$. In an orthonormal basis these become antisymmetric matrices, and do not depend on the choice of simple coroots or weights. Since they relate two different vector spaces, they can be thought of as unit matrices. Such a transformation maps the electric charges of the fundamental representation of $SU(3)$ on coroots, so we do not find support for duality under this $\mathbb{Z}_2$ group. In [37], subgroups of $Sp(r;\mathbb{Z})$ are considered that preserve the scalar products between roots of $SU(N_c)$ (up to a scale), so that the transformation of the "coupling matrix" only consists of a transformation of the complex coupling constant. We hope to return to a closer examination of subgroups that might explain parts of the spectrum we observe (though it is difficult to conceive how the entire spectra could be generated). Our attention has also been drawn to reference [39], where some of the arguments and results are very close to ours.

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Appendix A: Topology of the Moduli Space at $k=\alpha_1+\alpha_2$

As we have seen, the moduli space at $k=\alpha_1+\alpha_2$ is actually completely determined just by considering its isometries together with the hyperKähler property. In this appendix, we will use the correspondence between moduli spaces and spaces of rational holomorphic maps to get direct information about the topology of this space, and support the indirect arguments. This procedure could in principle be continued along the lines of [40] to obtain also the metric.

For $SU(2)$ monopoles, there is an isomorphism between the moduli space at charge k and the space of rational holomorphic maps $S^2 \rightarrow S^2$, due to Donaldson [41]. This result was extended to more general groups by Hurtubise [42], where the case of maximal breaking was considered, and the moduli spaces shown to be isomorphic to spaces of rational holomorphic maps from S^2 to G/H ("the broken gauge group"). The target space of the holomorphic map is a "flag manifold", *i.e.* a space of nested vector subspaces $\mathbb{C}\subset\mathbb{C}^2\subset\ldots\subset\mathbb{C}^N$. This makes it quite straightforward to write down explicit coordinates for these manifolds as $\mathbb{C}P^1$ bundles over $\mathbb{C}P^2$ bundles over ... over $\mathbb{C}P^{N-1}$.

We will examine the case of $SU(3)/(U(1)\times U(1))$, *i.e.* the manifold of complex lines in a complex plane in $\mathbb{C}^3$. This clearly implies an S^2 bundle over $\mathbb{C}P^2$. Explicit parametrization of the plane and the line, and some minor redefinition in order to make things as symmetric as possible, gives the coordinates $(x_i, y_i; \zeta_i)$ in patch i, $i=1,2,3$, with the overlaps

$$(x_{i+1}, y_{i+1}; \zeta_{i+1}) = (y_i^{-1}, x_i y_i^{-1}; -\alpha x_i - \alpha^{-1} y_i \zeta_i^{-1}) \ , \tag{A.1}$$

where $\alpha = e^{\frac{2\pi i}{3}}$ and $3+1$ is understood as 1. Here, the ζ coordinates are fiber coordinates for S^2. We have not cared to write two separate patches for the fiber, since the one-point compactification of $\mathbb{C}$ is trivial. The coordinates for the base manifold are the standard ones on $\mathbb{C}P^2$. If one instead considers the flag "turned inside out", *i.e.* consider the complementary (normal) vector subspaces, one is led to an alternative fibration, given by the coordinate transformations

$$(\tilde{x}_i, \tilde{y}_i; \tilde{\zeta}_i) = (\zeta_{i+1}^{-1}, \zeta_{i+2}; y_{i+1}) \ . \tag{A.2}$$

These coordinates have identical overlap relations as the original ones. The transformation corresponds to the action of the nontrivial element in the $\mathbb{Z}_2$ of outer automorphisms of $SU(3)$.

We now want to examine some simple rational holomorphic maps from S^2 to this manifold. These maps should be "based". We choose the base point condition $(x,y;\zeta)(\infty)=(1,1;1)$, which is the same in all patches. It is easy to find a basis for the second homotopy. The fiber S^2 of course has second homotopy $\mathbb{Z}$, and so has the base manifold $\mathbb{C}P^2$, being $S^5/U(1)$. The holomorphic maps corresponding to one winding on the fiber, *i.e.* one of the simple magnetic charges, say α_1, are easily written down:

$$\begin{pmatrix} x_1 & y_1 & \zeta_1 \\ x_2 & y_2 & \zeta_2 \\ x_3 & y_3 & \zeta_3 \end{pmatrix}(z) = \begin{pmatrix} 1 & 1 & \frac{z+A}{z+B} \\ 1 & 1 & \frac{z+B}{z+C} \\ 1 & 1 & \frac{z+C}{z+A} \end{pmatrix} \ , \tag{A.3}$$

where $A+\alpha B+\alpha^2 C=0$. The easiest way of finding the maps corresponding to the other simple root α_2 is to apply the coordinate transformation (A.2) to the right hand side of (A.3) to obtain

$$\begin{pmatrix} x_1 & y_1 & \zeta_1 \\ x_2 & y_2 & \zeta_2 \\ x_3 & y_3 & \zeta_3 \end{pmatrix}(z) = \begin{pmatrix} \frac{z+A}{z+C} & \frac{z+B}{z+C} & 1 \\ \frac{z+C}{z+B} & \frac{z+A}{z+B} & 1 \\ \frac{z+B}{z+A} & \frac{z+C}{z+A} & 1 \end{pmatrix} \ . \tag{A.4}$$

The corresponding monopoles are the embedded 't Hooft–Polyakov solutions, and it is easy to deduce that the topology of these moduli spaces is $\mathbb{C}\times\mathbb{C}^* \cong \mathbb{R}^3\times S^1$. A more interesting case is the magnetic charge $\alpha_1+\alpha_2$. This map winds once around each of the primitive cycles. We write down the most general ansatz possible, and then derive constraints on the parameters that enter:

$$\begin{pmatrix} x_1 & y_1 & \zeta_1 \\ x_2 & y_2 & \zeta_2 \\ x_3 & y_3 & \zeta_3 \end{pmatrix}(z) = \begin{pmatrix} \frac{z+A}{z+C} & \frac{z+B}{z+C} & \frac{z+D}{z+E} \\ \frac{z+C}{z+B} & \frac{z+A}{z+B} & \frac{z+F}{z+D} \\ \frac{z+B}{z+A} & \frac{z+C}{z+A} & \frac{z+E}{z+F} \end{pmatrix} . \tag{A.5}$$

The outer automorphisms act as $(A,B,C)\leftrightarrow(D,E,F)$. Using the overlap functions we arrive at the constraints between the six complex parameters:

$$\begin{aligned} &A+D+\alpha(B+E)+\alpha^2(C+F)=0 \ , \\ &AD+\alpha BE+\alpha^2 CF=0 \ , \end{aligned} \tag{A.6}$$

so that we arrive at the counting of section 4 for the dimension of this moduli space — it has real dimension eight.

When we investigate the topology, it is useful to consider holomorphic vector fields on the flag manifold. Some of these will generate holomorphic isometries on the moduli space. The regular vector fields we consider take the same form in all three patches (they are the only ones with this property):

$$\begin{aligned} \mathscr{V}^{(1)} &= (1-xy)\frac{\partial}{\partial x}+(x-y^2)\frac{\partial}{\partial y}-(\alpha+y\zeta+\alpha^{-1}x\zeta^2)\frac{\partial}{\partial\zeta} \ , \\ \mathscr{V}^{(2)} &= (y-x^2)\frac{\partial}{\partial x}+(1-xy)\frac{\partial}{\partial y}+(\alpha y+x\zeta+\alpha^{-1}\zeta^2)\frac{\partial}{\partial\zeta} \ . \end{aligned} \tag{A.7}$$

There are also the translations on S^2, inducing the vector field $\mathscr{V}^{(3)}=x'(z)\frac{\partial}{\partial x}+y'(z)\frac{\partial}{\partial y}+\zeta'(z)\frac{\partial}{\partial\zeta}$. All of these transformations commute. The transformations induce transformations of the parameters $A,\ldots,F$. These are better expressed in a basis where the vector fields act diagonally,

$$\begin{aligned} a&=A+B+C \ , & d&=D+E+F \ , \\ b&=A+\alpha B+\alpha^2 C \ , & e&=D+\alpha E+\alpha^2 F \ , \\ c&=A+\alpha^2 B+\alpha C \ , & f&=D+\alpha^2 E+\alpha F \ . \end{aligned} \tag{A.8}$$

Then $\mathscr{V}^{(3)}$ only acts on a and d as translation, while, if we denote the induced action of $\frac{i}{\sqrt{3}}(\alpha\mathscr{V}^{(1)}-\alpha^{-1}\mathscr{V}^{(2)})$ by δ_+ and that of $\frac{1}{3}(\alpha\mathscr{V}^{(1)}+\alpha^{-1}\mathscr{V}^{(2)})$ by δ_-, the action on the moduli parameters is

$$\begin{aligned} \delta_+ b&=2b \ , & \delta_+ e&=2e \ , \\ \delta_+ c&=c \ , & \delta_+ f&=f \ , \\ \\ \delta_- b&=0 \ , & \delta_- e&=0 \ , \\ \delta_- c&=c \ , & \delta_- f&=-f \ , \end{aligned} \tag{A.9}$$

while a and d are inert. The constraints are

$$\begin{aligned} &b+e=0 \ , \\ &ae+bd+cf=0 \ , \end{aligned} \tag{A.10}$$

and they are preserved by all the transformations. The transformation δ_+ generates the $\mathbb{C}^*$ that together with the $\mathbb{C}$ of $\mathscr{V}^{(3)}$ forms $\mathbb{R}^3 \times S^1$. The imaginary part of δ_- is a $U(1)$ isometry. We can chose a location θ on the S^1 by a finite action $\exp(i\theta Im\,\delta_+)$ on some given base point. The parameters c and f are coordinates for the "inner part" of the moduli space. By considering the action of this translation on the total moduli space, we conclude that the topology is

$$\mathscr{M} \cong \mathbb{R}^3 \times \frac{S^1 \times \mathbb{R}^4}{\mathbb{Z}_2} \ . \tag{A.11}$$

The "inner" or "relative" moduli space is topologically $\mathbb{R}^4$. This is the topology of Taub–NUT space with positive mass parameter.

Appendix B: Taub–NUT Space — Metric and Connections

This appendix contains a short summary about Taub–NUT space (see *e.g.* reference [43] for more detailed discussions). Taub–NUT space is a member of a very restricted family of four-dimensional regular hyperKähler manifolds with $SO(3)$ isometry [43], that also includes the Atiyah–Hitchin manifold (contained in the moduli space for magnetic charge twice a simple coroot), and the Eguchi–Hansson manifold. The properties obtained from simple physical considerations, that the metric asymptotically approaches $\mathbb{R}^3 \times S^1$ and that the isometry is $SU(2) \times U(1)$, singles out Taub–NUT as the internal moduli space for magnetic charges that are the sum of two simple coroots with negative scalar product.

The metric may be written

$$g = \frac{r+M}{r-M}\, dr \otimes dr + (r^2 - M^2)(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) + 4M^2\, \frac{r-M}{r+M}\, \sigma_3 \otimes \sigma_3 \ , \tag{B.1}$$

where the ranges of the coordinates are $M \leq r$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$ and $0 \leq \psi < 4\pi$, and the σ_i are left-invariant one-forms on $S^3 \cong SU(2)$:

$$\begin{aligned} \sigma_1 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi \ , \\ \sigma_2 &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi \ , \\ \sigma_3 &= d\psi + \cos\theta d\phi \ , \end{aligned} \tag{B.2}$$

with the dual vector fields v_i, $v_i(\sigma_j) = \delta_{ij}$:

$$\begin{aligned} v_1 &= \cos\psi \frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\phi} - \cot\theta \sin\psi \frac{\partial}{\partial\psi} \ , \\ v_2 &= -\sin\psi \frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\phi} - \cot\theta \cos\psi \frac{\partial}{\partial\psi} \ , \\ v_3 &= \frac{\partial}{\partial\psi} \ . \end{aligned} \tag{B.3}$$

If we write the vierbein one-forms as $e_r = f dr$, $e_i = c_i \sigma_i$, the functions f, c_i satisfy (prime denotes differentiation with respect to r)

$$\frac{c_1'}{f} = \frac{c_1^2 - (c_2 - c_3)^2}{2c_2 c_3} \qquad \text{and cyclic.} \tag{B.4}$$

This equation enables us to calculate the curvature quite easily:

$$\begin{aligned} R_{0i} &= \frac{1}{2}\varepsilon_{ijk}R_{jk} \\ R_{01} &= -k_1' dr\wedge\sigma_1 + (-k_1+k_2+k_3-2k_2k_3)\sigma_2\wedge\sigma_3 \qquad \text{and cyclic,} \end{aligned} \tag{B.5}$$

where $k_i=\frac{c_i'}{f}$. The first equation states that R is selfdual. The curvature may then be used in the calculation of the index of the Dirac operator [33], using the Atiyah–Patodi–Singer index theorem [44] and pushing the boundary to infinite radius.

When we consider matter zero-modes, we will need a U(1) connection on Taub–NUT space with selfdual field strength. There is exactly one selfdual harmonic two-form (up to normalization). It is

$$F = c\left(\frac{2M}{(r+M)^2}dr\wedge\sigma_3 - \frac{r-M}{r+M}\sigma_1\wedge\sigma_2\right) \ . \tag{8.1}$$

The corresponding potential is

$$\omega = c\,\frac{r-M}{r+M}\,\sigma_3 \ . \tag{8.2}$$

The coefficient c is determined by physical considerations.

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Superembeddings, Non-linear Supersymmetry and 5-branes

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Abstract

We examine general properties of superembeddings, i.e., embeddings of supermanifolds into supermanifolds. The connection between an embedding procedure and the method of non-linearly realised supersymmetry is clarified, and we demonstrate how the latter arises as a special case of the former. As an illustration, the super-5-brane in 7 dimensions, containing a self-dual 3-form world-volume field strength, is formulated in both languages, and provides an example of a model where the embedding condition does not suffice to put the theory on-shell.

1. Introduction

Our understanding of string theory at the non-perturbative level has gone through a dramatic improvement in recent years. Some of the key aspects of this development are connected to the central rôle played by solitonic solutions of the low energy field equations, i.e., various brane configurations that solve the field equations of the supergravity theories. By considering BPS saturated solitonic solutions which preserve e.g. half the supersymmetry of the supergravity theory in question, these supergravity theories can be shown to be related by duality transformations some of which are intrinsically non-perturbative in nature. In fact, (almost) all consistent string and supergravity theories, including 11-dimensional supergravity, are in this way believed to constitute low-energy descriptions of one master theory, the so called M-theory, in either the weak or strong coupling regime of some particular coupling constant in the moduli space of all couplings. An overview of the subject, as well as further references, may be found e.g. in ref. [1].

The known branes come in three main varieties[†], p-branes, Dp-branes, and T5-branes, depending on whether the bosonic sector of the field theory on the world-volume of the brane contains only scalars, scalars and vector gauge fields, or scalars together with a third rank anti-symmetric self-dual tensor field strength. (Recently also other types of tensor fields and combination of such have been introduced in these theories to solve certain specific problems [2]. However, this is of no immediate interest for the considerations of this paper). For a review of the different kinds of solitonic branes and their rôles in non-perturbative string theory, see ref. [3]. The scalar fields appearing on the branes are immediately identifyable as Goldstone fields, or collective modes, corresponding to the translation symmetries that are broken when the brane is introduced into target space-time. That is, one obtains one scalar field for each direction transverse to the brane. By checking which supersymmetries get broken, or by viewing the brane as a supersurface embedded in a target superspace, also the number of Goldstone fermions can be deduced. However, when supersymmetry requires the brane supermultiplet to contain also vectors or tensor potentials, there is no analogously simple argument that explains their presence. We will have nothing new to say about this problem in this paper.

From the theory of non-linear realisations (NR) we know that, although the branes fill out multiplets realising all target space symmetries linearly, on the branes the unbroken symmetries are linearly realised while the broken ones are realised non-linearly. In the context of open string theory one knows that the supersymmetric field theory on Dp-branes involve vector multiplets and are highly non-linear Born–Infeld type theories. Using duality arguments similar non-linearities can be seen to arise for T5-branes containing self-dual third rank tensors in $d = p + 1 = 6$ brane dimensions [4,5].

Bagger and Galperin [6] have recently verified that the theory of non-linear realisations applied to supermanifolds embedded into target supermanifolds with twice the number of anticommuting coordinates naturally leads to Born–Infeld actions if vector multiplets are involved. This provides

[†] There are also branes associated with gravitational charges. We will not consider these in the present paper.

a very nice explanation for the rather strange form of the Born–Infeld action as being a direct consequence of the non-linearly realised broken supersymmetries. In this formalism one introduces derivatives that transform in a well-behaved manner under the linearly as well as under the non-linearly realised (super)symmetries. Consistency requirements on the constraints imposed on superfields together with requirement of symmetry under the linearly as well as the non-linearly realised supersymmetry imply the Born–Infeld non-linear action in the case the supermultiplet is chosen to be a Maxwell multiplet in 4 dimensions.

Another recently developed approach giving similar results is the "doubly supersymmetric geometrical approach" [7,8] or the "embedding formalism" [9]. In the latter approach one starts from the torsion tensor in target superspace and considers the equation that arises when pulling it back to the super-world-volume. By introducing a particular embedding constraint the torsion pull-back equations can, in the only case analysed explicitly so far namely the T5-brane in 11 dimensions, be seen to give rise to exactly the same non-linear theory as can be argued for from its relation via duality to the Born–Infeld action of a D4-brane. However, in this formalism the non-linearly realised supersymmetry plays no rôle whatsoever, and it is not clear that the non-linearities of the action actually have their origin in some broken symmetries, although this clearly must be the case [10].

It is the purpose of this paper to clearify some aspects of the connection between these two approaches and demonstrate that also for the T5-brane the non-linearities of the action stem from an underlying set of broken symmetries. In section 2 we discuss some basic properties of superembeddings using as an example some results from the theory of non-linear realisations as well as from the theory of superembeddings applied to the T5-brane, with a $(6|8)$ super-world-volume, embedded into a $(7|16)$ target superspace. Here the notation $(m|n)$ refers to a superspace with m commuting and n anticommuting coordinates. Section 3 gives the details of this embedding using the theory of non-linear realisations along the lines of Bagger and Galperin [6]. This formalism turns out to generate a rather complicated equation that the dimension zero components of the torsion tensor induced on the super-world-volume must satisfy. Although this equation can be solved explicitly, further analysis of the system, e.g. deriving the field equations, seems cumbersome and is not carried out here. Instead we turn in the following sections to an analysis of this T5-brane by means of the embedding formalism. In section 4 we show that the theory of non-linear realisations is just a special case of the embedding formalism, obtained if certain for this formalism unconventional choices of intrinsic torsion components are made. In section 5 we then show that the torsion pull-back equation can be completely analysed and seen to lead to the non-linearities characteristic of T5-brane field theories, as already demonstrated for the T5-brane embedded in 11 dimensions by Howe, Sezgin and West [4]. In a final section we summarise our results and present the conclusions.

2. Superembeddings

In this section we will consider superembeddings [8,9] from a general point of view, using some explicit results from subsequent sections to examplify the ideas but leaving the details of special applications to the later sections. The different parametrisations of the embedding matrix to be used in later sections are introduced, and the geometric properties of the embeddings are analysed, eventually leading to the torsion pullback equation, introduces in ref. [9].

Let us consider an arbitrary embedding $(\mathscr{M},h) \overset{f}{\hookrightarrow} (\underline{\mathscr{M}},\underline{g})$, where the two supermanifolds have dimensions $(m|n)$ and $(\underline{m}|\underline{n})$ respectively. The signature of the bosonic metric is arbitrary at the moment but later on we will restrict ourselves to $(D-1,1)$ signature. We will use standard notation [9] for the local coordinates of the two supermanifolds, i.e., $z^M=(x^m;\theta^\mu)$ and $Z^{\underline{M}}=(X^{\underline{m}};\Theta^{\underline{\mu}})$.

We now introduce the embedding matrix* $\mathscr{E}_A{}^{\underline{A}}$, defined in terms of canonical 1-forms θ by

$$\tilde{\theta} := f_*\theta = f^*\underline{\theta} = e^A\mathscr{E}_A{}^{\underline{A}}E_{\underline{A}}\ . \tag{2.1}$$

Here, e^A and $E_{\underline{A}}$ are orthonormal basis vectors on the cotangent space of the world-volume and the tangent of the target space, respectively. We refer to Appendix A for more details of notation. The basis vectors $\mathscr{E}_A := f_*e_A$ span the tangent space of the embedded supermanifold. In order to have a complete basis for the entire tangent space of the target space, we may also introduce normal vectors denoted $\mathscr{E}_{A'}$. We will use an overlined index representing a composite index for the pair (A,A'). We will also introduce a set of dual basis vectors by

$$<\mathscr{E}_{\overline{B}},\mathscr{E}^{\overline{A}}>=\delta_{\overline{B}}{}^{\overline{A}}\ . \tag{2.2}$$

With these objects at hand we have the possibility of splitting the canonical 1-form into tangential and normal terms,

$$\underline{\theta}=\theta+\theta'\equiv\mathscr{E}^A\mathscr{E}_A+\mathscr{E}^{A'}\mathscr{E}_{A'}\ . \tag{2.3}$$

These 1-forms now serve as projectors of vectors down to the tangent and normal parts respectively, i.e., $X^{\parallel}=\theta(X)$, $X^{\perp}=\theta'(X)$. By introducing a target space Lorentz matrix $u_{\overline{B}}{}^{\underline{A}}$ relating the basis $E_{\underline{A}}$ to a frame connected to the embedded surface, it is convenient to split the embedding matrix as

$$\mathscr{E}_A{}^{\underline{A}}=\mathscr{E}_A{}^{\overline{B}}u_{\overline{B}}{}^{\underline{A}}\ . \tag{2.4}$$

Concerning the basis $\mathscr{E}_{A'}$ of normal vectors, the choice is completely arbitrary and physically irrelevant, and it will soon be clear that in explicit parametrisations we can always choose them to be $\mathscr{E}_{A'}{}^{\underline{A}}=u_{A'}{}^{\underline{A}}$, i.e., as part of a Lorentz matrix.

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* Note the difference in notation compared to refs. [9,4], where the matrix $\mathscr{E}$ does not denote the embedding matrix, the latter being denoted $E_A{}^{\underline{A}}$.

As a starting point for a general superembedding, the orientation in target superspace of the super-world-volume tangent space is parametrised by a point in the super-grassmanian

$$\mathrm{SGr}[(m|n);(\underline{m}|\underline{n})] := \frac{\mathrm{OSp}(\underline{m}|\underline{n})}{\mathrm{OSp}(m|n)\times\mathrm{OSp}(\underline{m}-m|\underline{n}-n)}\ , \tag{2.5}$$

i.e., there are $m(\underline{m}-m)+n(\underline{n}-n)$ bosonic parameters and $m(\underline{n}-n)+n(\underline{m}-m)$ fermionic ones. One way of representing these degrees of freedom is to introduce the four fields

$$\begin{array}{lcl} m_a{}^{b'} & \leftrightarrow & m(\underline{m}-m)\ , \\ h_\alpha{}^{\beta'} & \leftrightarrow & n(\underline{n}-n)\ , \\ \chi_a{}^{\beta'} & \leftrightarrow & m(\underline{n}-n)\ , \\ \mathscr{E}_\alpha{}^{b'} & \leftrightarrow & n(\underline{m}-m)\ , \end{array} \tag{2.6}$$

and locally represent the embedding by

$$\mathscr{E}_A{}^{\underline{B}} = \mathscr{E}_A{}^{\overline{B}} u_{\overline{B}}{}^{\underline{B}} = \begin{pmatrix} u_a{}^{\underline{a}} + m_a{}^{b'} u_{b'}{}^{\underline{a}} & \chi_a{}^{\alpha'} u_{\alpha'}{}^{\underline{\alpha}} \\ \mathscr{E}_\alpha{}^{b'} u_{b'}{}^{\underline{b}} & u_\alpha{}^{\underline{\alpha}} + h_\alpha{}^{\beta'} u_{\beta'}{}^{\underline{\alpha}} \end{pmatrix}\ . \tag{2.7}$$

If we put this together with the normal vectors we get

$$\mathscr{E}_{\overline{A}}{}^{\overline{B}} = \begin{pmatrix} \begin{pmatrix} \delta_a{}^b & m_a{}^{b'} \\ 0 & \delta_{a'}{}^{b'} \end{pmatrix} & \begin{pmatrix} 0 & \chi_a{}^{\beta'} \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \mathscr{E}_\alpha{}^{b'} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \delta_\alpha{}^\beta & h_\alpha{}^{\beta'} \\ 0 & \delta_{\alpha'}{}^{\beta'} \end{pmatrix} \end{pmatrix}\ , \tag{2.8}$$

with the inverse

$$(\mathscr{E}^{-1})_{\overline{A}}{}^{\overline{B}} = \begin{pmatrix} \begin{pmatrix} \delta_a{}^b & -m_a{}^{b'} \\ 0 & \delta_{a'}{}^{b'} \end{pmatrix} & \begin{pmatrix} 0 & -\chi_a{}^{\beta'} \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -\mathscr{E}_\alpha{}^{b'} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \delta_\alpha{}^\beta & -h_\alpha{}^{\beta'} \\ 0 & \delta_{\alpha'}{}^{\beta'} \end{pmatrix} \end{pmatrix}\ . \tag{2.9}$$

We notice that the information of the embedding lies entirely in the matter fields, and that $u_{\overline{A}}{}^{\underline{B}}$ can be chosen arbitrarily. As we will see in the case of non-linear realisations in sections 3 and 4, they may for example be chosen to be just $\delta_{\overline{A}}{}^{\underline{B}}$.

In all applications we will choose the part of the embedding matrix not containing the fields of (2.6), i.e., the u's, to be part of a Lorentz matrix. This choice is always possible, recalling that the essential property of the embedding matrix is that it defines the orientation of the embedded hypersurface, so that different embedding matrices with identical span of the vectors $\mathscr{E}_A$ represent the same point in the grassmannian (2.5), and thus the same embedding. To put it concretely, this degree of arbitrariness in the embedding matrix is identified with the invariance of its definition (2.1) under

$$\begin{array}{rcl} e^A & \to & e^B M_B{}^A \ , \\ \mathscr{E}_A{}^{\underline{A}} & \to & (M^{-1})_A{}^B \mathscr{E}_B{}^{\underline{A}} \ , \end{array} \tag{2.10}$$

allowing us to go to a representation (2.7) with lorentzian u's.

The canonical 1-forms are now expressed in terms of the matter fields in the following way:

$$\begin{array}{rcl} \theta_\chi & = & \theta_0 + m' + \mathscr{E} + \chi + h \ , \\ \theta'_\chi & = & \theta'_0 - m' - \mathscr{E} - \chi - h \ , \end{array} \tag{2.11}$$

and if we define new vielbeins by $E_{\overline{A}} := u_{\overline{A}}{}^{\underline{B}} E_{\underline{B}}$, we see that

$$\begin{array}{rcl} m' & = & E^a m_a{}^{b'} E_{b'} \ , \\ \mathscr{E} & = & E^\alpha \mathscr{E}_\alpha{}^{b'} E_{b'} \ , \\ \chi & = & E^a \chi_a{}^{\beta'} E_{\beta'} \ , \\ h & = & E^\alpha h_\alpha{}^{\beta'} E_{\beta'} \ . \end{array} \tag{2.12}$$

An example of the present parametrisation of the embedding matrix is given by the NR case (section 4), where we work in a supersymmetric supermanifold with $n = \underline{n}/2$. There we will see that the fields of (2.6) are simply

$$\begin{array}{rcl} m_a{}^{b'} & = & \nabla_a \phi^{b'} \ , \\ \mathscr{E}_\alpha{}^{b'} & = & \nabla_\alpha \phi^{b'} - i(\Gamma^{b'}\psi)_\alpha \ , \\ \chi_a{}^{\beta'} & = & \nabla_a \psi^{\beta'} \ , \\ h_\alpha{}^{\beta'} & = & \nabla_\alpha \psi^{\beta'} \ , \end{array} \tag{2.13}$$

where the bosonic matter fields $\phi^{b'}$ are shifted [9] as

$$\phi^{b'} = x^{b'} + \tfrac{i}{2}\theta\Gamma^{b'}\psi \ . \tag{2.14}$$

We also see that on imposing the embedding condition [8,9]

$$\mathscr{E}_\alpha{}^{\underline{b}} = 0 \ , \tag{2.15}$$

(this condition, which is a basic geometric relation reducing the number of field components in the embedding formalism, will be more closely examined in section 4) we get a relation [9]

$$\psi^{\beta'} = -\tfrac{i}{\underline{m}-m}(\Gamma_{c'})^{\beta'\alpha}\nabla_\alpha\phi^{c'} \tag{2.16}$$

between the bosonic fields $\phi^{b'}$ and the fermionic $\psi^{\beta'}$, which are the matter fields containing the independent degrees of freedom of the embedding ($(\underline{m}-m)$ and $(\underline{n}-n)$ respectively). Of course, since these fields are both superfields, they contain in general too many physical degrees of freedom. This problem will be eliminated by analysing the torsion equation together with the embedding condition.

Returning to our study of the embedding matrix, we note that with the above parametrisation it is only lorentzian for all matter fields equal to zero. It is easy to convince oneself that the field ${m_a}^{b'}$ can always be rotated away by a (target space) Lorentz tranformation:

$$ {m_a}^b \tilde{u}_b{}^{\underline{c}} := u_a{}^{\underline{c}} + {m_a}^{b'} u_{b'}{}^{\underline{c}} \ , \tag{2.17} $$

so that the $m(\underline{m}-m)$ parameters of the orientation of the bosonic embedding are absorbed into $\tilde{u}$. The price to be paid for this change of frame is that the fermions rotate accordingly, and the lower right hand corner of (2.7) changes. Again, it is possible to retain the form $u_\alpha{}^{\underline{\alpha}} + h_\alpha{}^{\beta'} u_{\beta'}{}^{\underline{\alpha}}$ by utilising the invariance (2.10) with a non-lorentzian matrix M. The embedding matrix then takes the form [9]

$$ \mathcal{E}_{\underline{A}}{}^{A} = \begin{pmatrix} \begin{pmatrix} {m_a}^b u_b{}^{\underline{a}} \\ u_{a'}{}^{\underline{a}} \end{pmatrix} & \begin{pmatrix} \chi_a{}^{\alpha'} u_{\alpha'}{}^{\underline{\alpha}} \\ 0 \end{pmatrix} \\ 0 & \begin{pmatrix} u_\alpha{}^{\underline{\alpha}} + h_\alpha{}^{\beta'} u_{\beta'}{}^{\underline{\alpha}} \\ u_{\alpha'}{}^{\underline{\alpha}} \end{pmatrix} \end{pmatrix} \ , \tag{2.18} $$

where the u's are again lorentzian (the tilde is dropped). This Lorentz matrix should of course not be identified with the one in (2.7), neither should the fields denoted by identical symbols. We have also dropped the $\mathcal{E}_\alpha{}^{\underline{a}}$ term as it will vanish due to the embedding condition. The new parametrisation also involves a new choice of basis for the normal vectors.

Equation (2.18) is the form of the embedding matrix to be used in the rest of the present section, and in section 5. The invariance (2.10) used to move between the two versions (2.7) and (2.18) of the embedding matrix involves a redefinition of the intrinsic vielbeins e^A, and we may expect the torsion tensors in the two versions of the theory to exhibit differences, which is what we will see in the following sections. It is striking that the seemingly different theories, from a geometric point of view, are related by a transformation that modifies the intrinsic world-volume geometry by matter fields. We will not analyse the transformations in detail, but note that they may be worth further study.

The inverse of the modified embedding matrix is

$$ \mathcal{E}_{\underline{A}}{}^{\overline{A}} = \begin{pmatrix} \left(u_{\underline{a}}{}^b (m^{-1})_b{}^a \ , \ u_{\underline{a}}{}^{a'} \right) & \left(0 \ , \ -u_{\underline{a}}{}^b (m^{-1})_b{}^a \chi_a{}^{\alpha'} \right) \\ 0 & \left(u_{\underline{\alpha}}{}^{\alpha} \ , \ u_{\underline{\alpha}}{}^{\alpha'} - u_{\underline{\alpha}}{}^{\beta} h_\beta{}^{\alpha'} \right) \end{pmatrix} \ , \tag{2.19} $$

and the canonical 1-forms therefore take the form

$$\begin{array}{rcl}\theta_\chi &=& \theta_0+\chi+h\ ,\\ \theta'_\chi &=& \theta'_0-\chi-h\ ,\end{array}\qquad(2.20)$$

where

$$\begin{array}{rcl}\chi &=& E^a(m^{-1})_a{}^b\chi_b{}^{\gamma'}E_{\gamma'}\ ,\\ h &=& E^\alpha h_\alpha{}^{\beta'}E_{\beta'}\ .\end{array}\qquad(2.21)$$

Here one should also mention that none of the free parameters in $m_a{}^{b'}$ ends up in $m_a{}^b$; the latter becomes determined completely in terms of h.

This is all we will say at this point about the parametrisation of the embedding matrix. We will now discuss the origin of the torsion pull-back equation and in later sections look at some of its solutions. To facilitate the understanding of the torsion equation we will point out some conceptual difficulties that may appear in connection with it. One problem is that when working with an embedding of the type $(\mathscr{M},h)\overset{f}{\hookrightarrow}(\underline{\mathscr{M}},\underline{g})$ we have to consider two different metrics on $\mathscr{M}$: on the one hand the a priori (intrinsic) metric on the world-volume h and on the other the metric induced by the embedding, $g=f^*\underline{g}$. The problem is that we no longer have one connection on $\mathscr{M}$ but two, each compatible with one distinct metric. We will use the notation $\mathscr{D}$ and ∇, schematically fulfilling

$$\begin{array}{rcl}\mathscr{D}h &=& 0\ ,\\ \nabla g &=& 0\ .\end{array}\qquad(2.22)$$

Another upcoming problem is connected to the fact that the embedding is not Lorentz, unless the matter fields vanish. In order to distinguish the situations, we will denote the matter fields collectively by χ, and let a lorentzian embedding correspond to $\chi\to0$.

In deriving the torsion equation, we start from the Gauss–Weingarten equations† [11]

$$\begin{array}{rcl}\underline{\nabla}_XY &=& \nabla_XY+\mathscr{K}'(X,Y)\ ,\\ \underline{\nabla}_XY' &=& \nabla'_XY'+\mathscr{K}(X,Y')\ ,\end{array}\qquad(2.23)$$

where we have used the notation X for tangential vectors and X' for normal vectors. We see from these equations that the covariant derivative splits into a tangential derivative, a normal derivative and two tensors which are the so called extrinsic curvatures of the embedding, also known as the second fundamental form*. These equations are purely tensorial and independent of the form of the embedding. They are also independent of the intrinsic metric h on the world-volume. If we now set $Y=E_A$ we get

$$\underline{\nabla}E_{\underline{A}}=:\Omega_{\underline{A}}{}^{\underline{B}}E_{\underline{B}}=\begin{pmatrix}\Omega_A{}^B & K_A{}^{B'}\\ K_{A'}{}^B & \Omega_{A'}{}^{B'}\end{pmatrix}\begin{pmatrix}E_B\\ E_{B'}\end{pmatrix}\ .\qquad(2.24)$$

The reason for taking $E_{\underline{A}}$ here instead of $\mathscr{E}_{\underline{A}}$ is that we need to make a distinction between whether the embedding is Lorentz or not. If the embedding is Lorentz then all quantities in these equations

† Ref. [9] gives similar equations, that in addition to our terms on the right hand side also contains the entities $\mathscr{L}$, $\mathscr{L}'$ which will soon be defined. The difference, as will be clear from the following discussion, resides entirely in the use of induced contra intrinsic connection in the derivative.

* The term is reserved for $\mathscr{K}'$, but $\mathscr{K}$ is determined from it by $\underline{g}(\mathscr{K}'(X,Y),Z')+\underline{g}(X,\mathscr{K}(Y,Z'))=0$.

will lie in the algebra *spin*($\underline{m}$) but not otherwise. We will therefore make a distinction between the extrinsic curvatures of the two types of embeddings by denoting the extrinsic curvature of a Lorentz embedding by roman letters and a matter triggered embedding by calligraphic ones. From the Gauss–Weingarten equations it follows that

$$\mathscr{K}_{AB}{}^{C'} = < \underline{\nabla}_A(\mathscr{E}_B), \mathscr{E}^{C'} > \ . \tag{2.25}$$

This means that, as the $\mathscr{E}_{\overline{A}}$ tend to $E_{\overline{A}}$ as $\chi \to 0$, the extrinsic curvature will tend to the Lorentz one, i.e., $\mathscr{K}_{AB}{}^{C'}|_{\chi=0} = K_{AB}{}^{C'}$. Of course

$$K_{AB}{}^{C'} = < \underline{\nabla}_A(E_B), E^{C'} > = < \underline{\nabla}_A(u_B), u^{C'} > \ , \tag{2.26}$$

where $u_B = u_A{}^{\underline{A}} E_{\underline{A}}$ and $u^{A'} = E^{\underline{A}} u_{\underline{A}}{}^{A'}$. Now since we will use the intrinsic world-volume metric h as an auxiliary field in the forthcoming torsion equation, we need a relation between the two connections on $\mathscr{M}$. Let us define a difference operator of the two of them by

$$\mathscr{L} := \nabla - \mathscr{D} \ . \tag{2.27}$$

This operator is of course a tensor. Proceeding as for the extrinsic curvature we let $\mathscr{L}|_{\chi=0} =: L$. We will also extend our covariant derivatives on $\mathscr{M}$ to act on world-volume vectors as well as target space vectors and denote them as $\overline{\nabla}$ and $\overline{\mathscr{D}}$. This enables us to note the following important relations

$$\begin{array}{rcl} \overline{\nabla}(\tilde{\theta}) & = & \mathscr{K}' \ , \\ \overline{\mathscr{D}}(\tilde{\theta}) & = & \mathscr{L} + \mathscr{K}' \end{array} \tag{2.28}$$

(these are tensor equations, so there is no wedge product involved), from which we see that the tensors can be written

$$\begin{array}{rcl} \mathscr{L}_{AB}{}^{C} & = & \overline{\mathscr{D}}_A(\mathscr{E}_B{}^{\underline{C}})\mathscr{E}_{\underline{C}}{}^{C} \ , \\ \mathscr{K}_{AB}{}^{C'} & = & \overline{\mathscr{D}}_A(\mathscr{E}_B{}^{\underline{C}})\mathscr{E}_{\underline{C}}{}^{C'} \ , \end{array} \tag{2.29}$$

and consequently

$$\begin{array}{rcl} L_{AB}{}^{C} & = & \overline{\mathscr{D}}_A(u_B{}^{\underline{C}})u_{\underline{C}}{}^{C} \ , \\ K_{AB}{}^{C'} & = & \overline{\mathscr{D}}_A(u_B{}^{\underline{C}})u_{\underline{C}}{}^{C'} \ . \end{array} \tag{2.30}$$

Let us introduce yet another covariant derivative in order to get relations between these fields:

$$\hat{\overline{\mathscr{D}}} = \overline{\mathscr{D}}|_{\mathrm{diag}} + \hat{X} \ , \tag{2.31}$$

where

$$\hat{X}_{\overline{A}}{}^{\overline{B}} := \begin{pmatrix} L_A{}^B & 0 \\ 0 & L_{A'}{}^{B'} \end{pmatrix} \ , \tag{2.32}$$

and where the connection in the first term on the right hand side contains the target space connection projected on the part not mixing tangential and normal directions. Let us also define

$$\mathscr{X}_{\underline{A}}{}^{\overline{B}} := \overline{\mathscr{D}}(\mathscr{E}_{\underline{A}}{}^{\underline{C}})\mathscr{E}_{\underline{C}}{}^{\overline{B}} \equiv \begin{pmatrix} \mathscr{L}_A{}^B & \mathscr{K}_A{}^{B'} \\ \mathscr{K}_{A'}{}^B & \mathscr{L}_{A'}{}^{B'} \end{pmatrix} \ . \tag{2.33}$$

This notion is natural because it will tend to L and K as $\chi \to 0$. We now get the relation between the fields

$$\mathscr{X}_{\underline{A}}{}^{\overline{B}} = \hat{X}_{\underline{A}}{}^{\overline{B}} + \hat{\overline{\mathscr{D}}}(\mathscr{E}_{\underline{A}}{}^{\overline{C}})(\mathscr{E}^{-1})_{\overline{C}}{}^{\overline{B}} + \mathscr{E}_{\underline{A}}{}^C K_C{}^{D'}(\mathscr{E}^{-1})_{D'}{}^{\overline{B}} + \mathscr{E}_{\underline{A}}{}^{C'} K_{C'}{}^D(\mathscr{E}^{-1})_D{}^{\overline{B}} \ , \tag{2.34}$$

from which, if we look at our parametrisation of the embedding matrix in particular, we now get the following relations

$$\begin{array}{rcl}
\mathscr{L}_b{}^c &=& L_b{}^c + (\hat{\overline{\mathscr{D}}} m_b{}^d)(m^{-1})_d{}^c \ , \\
\mathscr{L}_b{}^\gamma &=& \chi_b{}^{\beta'} K_{\beta'}{}^\gamma \ , \\
\mathscr{L}_\beta{}^c &=& 0 \ , \\
\mathscr{L}_\beta{}^\gamma &=& L_\beta{}^\gamma + h_\beta{}^{\beta'} K_{\beta'}{}^\gamma \ , \\
\mathscr{K}_b{}^{c'} &=& m_b{}^c K_c{}^{c'} \ , \\
\mathscr{K}_b{}^{\gamma'} &=& \hat{\overline{\mathscr{D}}}\chi_b{}^{\gamma'} - (\hat{\overline{\mathscr{D}}} m_b{}^c)(m^{-1})_c{}^d \chi_d{}^{\gamma'} - \chi_b{}^{\beta'} K_{\beta'}{}^\gamma h_\gamma{}^{\gamma'} \ , \\
\mathscr{K}_\beta{}^{c'} &=& 0 \ , \\
\mathscr{K}_\beta{}^{\gamma'} &=& K_\beta{}^{\gamma'} + \hat{\overline{\mathscr{D}}} h_\beta{}^{\gamma'} - h_\beta{}^{\beta'} K_{\beta'}{}^\gamma h_\gamma{}^{\gamma'} \ , \\
\mathscr{K}_{b'}{}^c &=& K_{b'}{}^d (m^{-1})_d{}^c \ , \\
\mathscr{K}_{b'}{}^\gamma &=& 0 \ , \\
\mathscr{K}_{\beta'}{}^c &=& 0 \ , \\
\mathscr{K}_{\beta'}{}^\gamma &=& K_{\beta'}{}^\gamma \ , \\
\mathscr{L}_{b'}{}^{c'} &=& L_{b'}{}^{c'} \ , \\
\mathscr{L}_{b'}{}^{\gamma'} &=& -K_{b'}{}^d (m^{-1})_d{}^e \chi_e{}^{\gamma'} \ , \\
\mathscr{L}_{\beta'}{}^{c'} &=& 0 \ , \\
\mathscr{L}_{\beta'}{}^{\gamma'} &=& L_{\beta'}{}^{\gamma'} - K_{\beta'}{}^\gamma h_\gamma{}^{\gamma'} \ .
\end{array} \tag{2.35}$$

Some of the zeroes are directly related to the embedding condition (2.15). The virtue of these relations is that they display explicitly which properties of the geometry are induced by matter fields. They are important because we will use them in the process of solving the torsion equation. We now turn to the issue of deriving the torsion equation, which is the final subject of this section.

If we look at the Gauss–Weingarten equations we see that

$$\underline{T}(X,Y) := \underline{\nabla}_X Y - \underline{\nabla}_Y X - [X,Y] = T(X,Y) + T'(X,Y) \ , \tag{2.36}$$

where $T(X,Y)$ is the induced torsion inherited from the connection on $T\underline{\mathscr{M}}$ and

$$T'(X,Y) := \mathscr{K}'(X,Y) - \mathscr{K}'(Y,X) \tag{2.37}$$

is called the extrinsic torsion of the embedding. But we know that we have a relation between the induced torsion and the intrinsic torsion denoted $\mathscr{T}$ from the relation of the two connections on $\mathscr{M}$. This relation is

$$T(X,Y) = \mathscr{T}(X,Y) + \mathscr{L}(X,Y) - \mathscr{L}(Y,X)\ , \tag{2.38}$$

which together with the relation

$$\overline{\mathscr{D}} \wedge \tilde{\theta} = \wedge\mathscr{L} + T' = -\mathscr{T} + T + T' \tag{2.39}$$

(the notation $\wedge\mathscr{L}$ meaning the antisymmetric part) finally yields the torsion equation in the form

$$\overline{\mathscr{D}} \wedge \tilde{\theta}(X,Y) + \mathscr{T}(X,Y) = \underline{T}(X,Y)\ , \tag{2.40}$$

where of course X, Y everywhere are super-world-volume tangent vectors. This is nothing but the usual torsion equation that figures in the physics literature [8,9]. Putting $X = \mathscr{E}_A$ and $Y = \mathscr{E}_B$ and contracting with $E^{\underline{C}}$ we get it in the more transparent form

$$\overline{\mathscr{D}}_A \mathscr{E}_B{}^{\underline{C}} - (-1)^{AB} \overline{\mathscr{D}}_B \mathscr{E}_A{}^{\underline{C}} + \mathscr{T}_{AB}{}^C \mathscr{E}_C{}^{\underline{C}} = (-)^{A(B+\underline{B})} \mathscr{E}_B{}^{\underline{B}} \mathscr{E}_A{}^{\underline{A}} T_{\underline{AB}}{}^{\underline{C}}\ . \tag{2.41}$$

In order to solve this equation we will project it onto the tangent and the normal directions, respectively, giving

$$2\mathscr{L}_{[AB)}{}^C + \mathscr{T}_{AB}{}^C = T_{AB}{}^C \tag{2.42}$$

and

$$2\mathscr{K}_{[AB)}{}^{C'} = T_{AB}{}^{C'}\ , \tag{2.43}$$

where the graded anti-symmetrisation is defined by $V_{[AB)} := \frac{1}{2}(V_{AB} - (-1)^{AB} V_{BA})$. Now for the parametrisation in eq. (2.18) we have the following induced torsion components (if the target space is flat):

$$
\begin{array}{rcl}
T_{ab}{}^{c} &=& i[\chi_a(\Gamma^d)\chi_b](m^{-1})_d{}^c \ , \\
T_{ab}{}^{\gamma} &=& 0 \ , \\
T_{a\beta}{}^{c} &=& -i[\chi_a(\Gamma^d)h_\beta](m^{-1})_d{}^c \ , \\
T_{a\beta}{}^{\gamma} &=& 0 \ , \\
T_{\alpha\beta}{}^{c} &=& -i[(\Gamma^d)_{\alpha\beta}+h_\alpha(\Gamma^d)h_\beta](m^{-1})_d{}^c \ , \\
T_{\alpha\beta}{}^{\gamma} &=& 0 \ , \\
T_{ab}{}^{c'} &=& 0 \ , \\
T_{ab}{}^{\gamma'} &=& -i[\chi_a(\Gamma^d)\chi_b](m^{-1})_d{}^e\chi_e{}^{\gamma'} \ , \\
T_{a\beta}{}^{c'} &=& -i\chi_a(\Gamma^{c'})_\beta \ , \\
T_{a\beta}{}^{\gamma'} &=& i[\chi_a(\Gamma^d)h_\beta](m^{-1})_d{}^e\chi_e{}^{\gamma'} \ , \\
T_{\alpha\beta}{}^{c'} &=& -i2h_{(\alpha}(\Gamma^{c'})_{\beta)} \ , \\
T_{\alpha\beta}{}^{\gamma'} &=& i[(\Gamma^d)_{\alpha\beta}+h_\alpha(\Gamma^d)h_\beta](m^{-1})_d{}^e\chi_e{}^{\gamma'} \ .
\end{array}
\tag{2.44}
$$

The Γ matrices have been split according to appendix B, and summed α' indices are suppressed, e.g. $h_\alpha(\Gamma^d)h_\beta \equiv h_\alpha{}^{\alpha'}(\bar{\Gamma}^d)_{\alpha'\beta'}h_\beta{}^{\beta'}$. Together with the expressions for the fields $\mathscr{K}$, $\mathscr{L}$ and of course $\mathscr{T}$ it is just to begin solving for the matter fields. We already here see that the solutions will depend on the chosen intrinsic world-volume torsion $\mathscr{T}$, but we will come back to this in later sections. If we instead look at the case of our first parametrisation, given in eq. (2.7), where we had a direct coupling to the NR case, we get

$$
\begin{array}{rcl}
T_{ab}{}^{c} &=& i\chi_a(\Gamma^c)\chi_b \ , \\
T_{ab}{}^{\gamma} &=& 0 \ , \\
T_{a\beta}{}^{c} &=& -i\chi_a(\Gamma^c)h_\beta \ , \\
T_{a\beta}{}^{\gamma} &=& 0 \ , \\
T_{\alpha\beta}{}^{c} &=& -i[(\Gamma^c)_{\alpha\beta}+h_\alpha(\Gamma^c)h_\beta] \ , \\
T_{\alpha\beta}{}^{\gamma} &=& 0
\end{array}
\tag{2.45}
$$

(again, although the fields denoted by the same letters in (2.44) and (2.45) are related by field redefinitions, they should by no means be identified), which we will see in later sections is nothing but the relations for the torsion derived from the algebra of the induced covariant derivatives.

3. The $D=6$ tensor multiplet and non-linear realisations

In this section we will review the basic steps of the theory of non-linear realisations [12], which is a systematic way of studying the properties of Goldstone fields. It is well-known that the spontaneous breaking of supersymmetry gives rise to a massless spin-$\frac{1}{2}$ Goldstone fermion [13]. This fermion then belongs to the massless multiplet of the residual unbroken supersymmetry. However, the choice of the Goldstone multiplet is not unique. The partial breaking of $N=2$ supersymmetry to $N=1$ in four dimensions was studied in [6], for three different multiplets. We will use non-linear realisations to describe the spontaneous breaking of $N=1$ supersymmetry in $D=7$ to $N=(1,0)$ in $D=6$ and pick the self-dual tensor multiplet in 6 dimensions [14,15] as the Goldstone multiplet.

Let $\underline{\mathscr{M}}^{(7|16)}$ be a flat $N=1$ target superspace with local coordinates $Z^{\underline{M}}=(X^{\underline{m}},\Theta^{\underline{\mu}})$. Our starting point is the 7-dimensional $N=1$ supersymmetry algebra

$$\{Q_{\underline{\alpha}},Q_{\underline{\beta}}\}=(\Gamma^{\underline{a}})_{\underline{\alpha\beta}}P_{\underline{a}}\ . \tag{3.1}$$

Making the 7→6+1 split, using the conventions of appendix B, this algebra reads:

$$\begin{array}{rcl}\{Q^i_\alpha,Q^j_\beta\} &=& \varepsilon^{ij}(\gamma^a)_{\alpha\beta}P_a\ ,\\ \{Q^i_\alpha,S^\beta_j\} &=& \delta_\alpha{}^\beta\delta^i{}_jZ\ ,\\ \{S^\alpha_i,S^\beta_j\} &=& \varepsilon_{ij}(\gamma^a)^{\alpha\beta}P_a\ .\end{array} \tag{3.2}$$

Here ε^{ij} is the invariant tensor of the SU(2) automorphism group. From a 6-dimensional point of view, this is an $N=(1,1)$ algebra with a central charge Z, the momentum in the seventh direction. We now consider the partial breaking of this $N=(1,1)$ algebra down to $N=(1,0)$. Let Q^i_α be the unbroken $N=1$ supersymmetry generator and S^α_i its broken counterpart. A parametrisation of the $N=1$ target superspace $\underline{\mathscr{M}}^{(7|16)}$ suitable for our problem is

$$\Omega=\exp[i(x^aP_a+\theta^\alpha_iQ^i_\alpha)]\exp[i(yZ+\psi^i_\alpha S^\alpha_i)]\ . \tag{3.3}$$

Now the spinor $\psi^i_\alpha=\psi^i_\alpha(x,\theta)$ is the Goldstone superfield associated with the broken generator S^α_i, and the scalar $y=y(x,\theta)$ is the Goldstone superfield associated with the central charge Z. Here we have employed the "static gauge" for the splitting of target space coordinates:

$$\begin{array}{rclcrcl} X^m &=& x^m\ , && X^6 &=& y(x,\theta)\ ,\\ \Theta^\mu &=& \theta^\mu_i\ , && \Theta^{\mu'} &=& \psi^i_\mu(x,\theta)\ .\end{array} \tag{3.4}$$

Note that this construction naturally corresponds to the embedding $\mathscr{M}^{(6|8)}\hookrightarrow\underline{\mathscr{M}}^{(7|16)}$, where the Goldstone fields are bosonic and fermionic coordinates describing the shape of the supersurface $\mathscr{M}^{(6|8)}$, which automatically breaks half of the supersymmetry.

The S-supersymmetry acts with g=exp$(i\eta S)$ on Ω by left multiplication, $g\Omega=\Omega'$, which induces a transformation on the bosonic coordinates

$$\delta_\eta x^a=-\tfrac{i}{2}\eta\bar\gamma^a\psi\ . \tag{3.5}$$

This in turn makes the transformations of the Goldstone fields contain non-linear terms, in addition to the usual shifts:

$$\begin{aligned}\delta_\eta \psi^i_\alpha &= \eta^i_\alpha + \tfrac{i}{2}\eta\bar{\gamma}^a\psi\partial_a\psi^i_\alpha \ ,\\ \delta_\eta y &= -\tfrac{i}{2}\eta\theta + \tfrac{i}{2}\eta\bar{\gamma}^a\psi\partial_a y \ .\end{aligned} \tag{3.6}$$

Since the Cartan 1-form $\Omega^{-1}d\Omega$ takes its value in the supersymmetry algebra, we can parametrise it in the following way

$$\Omega^{-1}d\Omega = i[E^a P_a + E^6 Z + E^\alpha_i Q^i_\alpha + E^i_\alpha S^\alpha_i] \ . \tag{3.7}$$

This expansion gives the covariant world-volume Goldstone 1-forms:

$$\begin{aligned}E^a &= dx^a - \tfrac{i}{2}[d\theta\tilde{\gamma}^a\theta + d\psi\bar{\gamma}^a\psi] \ , \qquad & E^\alpha_i &= d\theta^\alpha_i \ ,\\ E^6 &= dy - \tfrac{i}{2}[d\theta\psi + d\psi\theta] \ , & E^i_\alpha &= d\psi^i_\alpha \ .\end{aligned} \tag{3.8}$$

Here we use the notation $\tilde{\gamma}^a := \epsilon^{ij}(\gamma^a)_{\alpha\beta}$ and $\bar{\gamma}^a := \epsilon_{ij}(\gamma^a)^{\alpha\beta}$. The vielbein matrix $E_M{}^A$ is found from the expansion of the world-volume 1-form $E^A = (E^a, E^\alpha_i)$ with respect to the coordinate differential $dz^M = (dx^m, d\theta^\mu_i)$ of the world-volume, $E^A = dz^M E_M{}^A$. The $N = 2$ derivatives induced by the Goldstone superfields are then given by†

$$\nabla_A = (E^{-1})_A{}^M\partial_M \ . \tag{3.9}$$

These covariant derivatives can be explicitly written as:

$$\begin{aligned}\nabla_a &= (E^{-1})_a{}^m\partial_m \ ,\\ \nabla^i_\alpha &= D^i_\alpha + \tfrac{i}{2}(D^i_\alpha\psi)\bar{\gamma}^a\psi\nabla_a \ .\end{aligned} \tag{3.10}$$

It is interesting to note that the covariant derivative ∇_a satisfies the implicit relation

$$\nabla_a = D_a + \tfrac{i}{2}(D_a\psi)\bar{\gamma}^a\psi\nabla_a \ , \tag{3.11}$$

which simply follows from solving for ∂_m above. This expression then most easily gives the expression for ∇^i_α above, which otherwise, when directly solved for as the dual of (3.8), is expressed in terms of the bare derivatives ∂_α. Here D^i_α and D_a are the ordinary flat $N = 1$ covariant derivatives. It is then straightforward to calculate the algebra of the $N = 2$ covariant derivatives [6]:

$$\begin{aligned}[\nabla_a, \nabla_b] &= -i(\nabla_a\psi)\bar{\gamma}^c(\nabla_b\psi)\nabla_c \ ,\\ [\nabla_a, \nabla^i_\alpha] &= i(\nabla^i_\alpha\psi)\bar{\gamma}^b(\nabla_a\psi)\nabla_b \ ,\\ \{\nabla^i_\alpha, \nabla^j_\beta\} &= i\epsilon^{ij}\gamma^a_{\alpha\beta}\nabla_a + i(\nabla^i_\alpha\psi)\bar{\gamma}^a(\nabla^j_\beta\psi)\nabla_a \ ,\end{aligned} \tag{3.12}$$

† These induced covariant derivatives, denoted ∇ in the present paper (see appendix A) equal those denoted $\mathscr{D}$ in ref. [6].

in accordance with eq. (2.45).

It is convenient to introduce the scalar superfield

$$\Phi := \tfrac{1}{2}\theta\psi - iy \ , \tag{3.13}$$

which, in particular, implies:

$$E^6 = id\Phi - id\theta\psi \ . \tag{3.14}$$

This shift, anticipated in section 2, is necessary in order to obtain a scalar superfield under the 6-dimensional supersymmetry algebra. Note that at this stage there is no relation between the Goldstone fields. We now impose the irreducibility condition

$$E_\alpha^{i6} = 0 \ , \tag{3.15}$$

or equivalently

$$\psi_\alpha^i = \nabla_\alpha^i \Phi \ . \tag{3.16}$$

In the next section we will see that this constraint is inherent in the embedding formalism, where it is part of the embedding condition ${\mathscr{E}_\alpha}^{\overline{a}} = 0$. In the present treatment its remaining components ${\mathscr{E}_\alpha}^a$ vanish trivially.

The on-shell self-dual tensor multiplet in 6 dimensions is given by

$$(1,0) \oplus 2(\tfrac{1}{2},0) \oplus (0,0) \ \leftrightarrow \ A^+_{ab} \oplus \psi_\alpha^i \oplus \phi \ , \tag{3.17}$$

where ψ_α^i and ϕ are the leading components of the spinor Goldstone superfield and the shifted scalar superfield, respectively, and where we have used the standard labeling of the massless particles by the helicity states of the little group Spin(4)$\approx$SU(2)$\times$SU(2). The minimal $N = (1,0)$ supersymmetry in 6 dimensions does indeed admit this tensor multiplet [14]. Here A is a 2-form potential coming from the symmetric bispinor superfield [15]

$$F_{\alpha\beta} := \tfrac{1}{2}\nabla_{(\alpha i}\nabla^i_{\beta)}\Phi := \nabla_{\alpha\beta}\Phi \ , \tag{3.18}$$

which corresponds to a self-dual field strength $F_{\alpha\beta} = \frac{1}{6}(\Gamma^{abc})_{\alpha\beta}F_{abc}$. It has been suggested [6] that there might be an extension of the $N = 2$ supersymmetry which associates a Goldstone-like symmetry with this field and the tensor gauge field might itself be a Goldstone field.

To describe the on-shell self-dual tensor multiplet, the superfield Φ has to be further constrained. This constraint is most easily expressed in terms of the $N=2$ covariant derivatives. An appropriate constraint can be found from the decomposition* [15]

$$\nabla^i_\alpha \nabla^j_\beta \equiv -\tfrac{1}{2} T_{\alpha\beta}^{ija} \nabla_a + \epsilon^{ij} \nabla_{\alpha\beta} + \nabla^{(i}_{[\alpha} \nabla^{j)}_{\beta]} \ . \tag{3.19}$$

Let us first consider the linear case. This decomposition then reads

$$D^i_\alpha D^j_\beta \equiv \tfrac{i}{2} \epsilon^{ij} (\gamma^a)_{\alpha\beta} \partial_a + \epsilon^{ij} D_{\alpha\beta} + D^{(i}_{[\alpha} D^{j)}_{\beta]} \ , \tag{3.20}$$

since the representation $(\mathbf{10},\mathbf{3})$ vanishes, $D^{(i}_{(\alpha} D^{j)}_{\beta)} \equiv 0$. It is easily shown [15] that the constraint

$$D^{(i}_{[\alpha} D^{j)}_{\beta]} \Phi = 0 \ , \tag{3.21}$$

postulating the absence of fields in the representation $(\mathbf{6},\mathbf{3})$, describes the on-shell self-dual tensor multiplet.

Turning to the full non-linear case again, we make the assumption, later to be verified, that the constraint generalises as

$$\nabla^{(i}_{[\alpha} \nabla^{j)}_{\beta]} \Phi = 0 \ . \tag{3.22}$$

The world-volume torsion is given by the implicit equation

$$\{\nabla^i_\alpha, \nabla^j_\beta\} =: -T_{\alpha\beta}^{ij\,a} \nabla_a = i\epsilon^{ij} (\gamma^a)_{\alpha\beta} \nabla_a + i(\nabla^i_\alpha \psi) \bar{\gamma}^a (\nabla^j_\beta \psi) \nabla_a \ . \tag{3.23}$$

Note that this is a highly non-linear equation, since the fact that $\psi^i_\alpha = \nabla^i_\alpha \Phi$ implies that also the right hand side contains torsion. We now proceed to give an explicit expression for this component of the induced torsion on-shell. Using the constraint above and acting on the scalar superfield Φ, we get the torsion equation on the form

$$2T^{ij}_{\alpha\beta} = \gamma^{ij}_{\alpha\beta} + (T^{ik}_{\alpha\gamma} + \epsilon^{ik} F_{\alpha\gamma}) \gamma^{\gamma\delta}_{kl} (T^{jl}_{\beta\delta} + \epsilon^{jl} F_{\beta\delta}) \ , \tag{3.24}$$

where $T^{ij}_{\alpha\beta} := -\frac{1}{2} T_{\alpha\beta}^{ij\,a} \nabla_a \Phi$ and $\gamma^{ij}_{\alpha\beta} := i\epsilon^{ij} (\gamma^a)_{\alpha\beta} \nabla_a \Phi$. The crucial point is that the totally symmetric representation $(\mathbf{10},\mathbf{3})$ drops out of the torsion after the on-shell constraint is imposed, and therefore

$$T_{\alpha\beta}^{ija} = \epsilon^{ij} T_{\alpha\beta}{}^a \ . \tag{3.25}$$

* We label the irreducible parts of the decomposition as $(\mathbf{4},\mathbf{2}) \otimes (\mathbf{4},\mathbf{2}) \equiv (\mathbf{6},\mathbf{1}) \oplus (\mathbf{10},\mathbf{1}) \oplus (\mathbf{6},\mathbf{3}) \oplus (\mathbf{10},\mathbf{3})$, reflecting the group structure Spin(1,5)×SU(2).

The torsion equation can then be written as the matrix equation

$$2T = \gamma + (T + F)\gamma(F - T) \ , \tag{3.26}$$

by extracting an overall ϵ^{ij}. It is convenient to introduce a matrix A such that $A^2 := \gamma$. Then let $B := AFA$ and $X := ATA$. The torsion equation now reads

$$2X = A^4 + (X + B)(B - X) \ , \tag{3.27}$$

with the solution

$$X = -1 + \sqrt{(1 + A^4 + B^2)} \ . \tag{3.28}$$

Note that $A^4 = (\nabla\Phi)^2$. In the weak-field expansion we get

$$X = \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} A^{4n} + \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} (1 + A^4)^{\frac{1}{2} - 2n} B^{2n} \ . \tag{3.29}$$

The torsion is then explicitly given by

$$T = \tfrac{1}{2}\gamma + \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n+1} (\nabla\Phi)^2\gamma + \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} (1 + (\nabla\Phi)^2)^{2n-1} F(\gamma F)^{2n-1} \ . \tag{3.30}$$

It is essential for obtaining $T_{\alpha\beta}{}^a$ that it is possible to extract a factor $\nabla\Phi$. To check that the supersymmetry algebra closes on the self-dual tensor multiplet it is sufficient to calculate $\nabla_\alpha\nabla_\beta\nabla_\gamma\Phi$. This check is cumbersome due to the fact that $T_{a\alpha}{}^b$ is given by a linear equation which in turn depends on $T_{\alpha\beta}{}^a$. From the solution (3.30), we see that the supersymmetry transformations of the component fields will be extremely non-linear. However, no more fields are generated. Hence our $N = 2$ covariant constraint, eq. (3.22) is correct, and puts the theory on-shell.

We conclude that the $N = (1, 0)$ self-dual tensor multiplet in 6 dimensions can indeed be given an interpretation as a Goldstone multiplet for the chirally broken $N = (1, 1)$ (or, actually 7-dimensional) supersymmetry, which is natural from a brane viewpoint. Since there is no lagrangian formulation of the theory (without the introduction of auxiliary fields [16], which however do not seem to have any natural interpretation in the present framework), the program pursued for e.g. the Maxwell multiplet in ref. [6], where a lagrangian formulation was derived, has no counterpart for this supermultiplet. The constraint (3.22), which is the most naive covariantisation of the irreducibility constraint of the linear theory, turns out to be consistent, and encodes the full non-linear field equations. Due to the complicated nature of the torsion, given by an implicit relation

(3.24) solved as (3.30), the derivation of the field equations for the component fields becomes cumbersome, and will not be performed here. We note that the explicit form of the torsion may be summarised as a formal square root, an observation that probably is connected to the relation with Born–Infeld theory.

4. Non-linear realisations in the embedding formalism

In this section we review some of the salient features of the embedding formalism, as applied to the superembedding of the 5-brane in $D = 7$. The "embedding formalism" [9] or the "doubly supersymmetric approach" [7,8] to describe p-brane dynamics[†] are based on a geometrical condition specifying the superembedding of a world-volume into target space. This condition can furthermore be obtained from a "generalised geometrical action principle" [7]. The power of the formalism was demonstrated in [4] for the T5-brane in 11 dimensions, where the embedding condition was postulated and supersymmetric equations of motion obtained before a complete supersymmetric action for them was constructed [5].

Consider the flat target superspace $\underline{\mathscr{M}}^{(7|16)}$ locally parametrised with coordinates $Z^{\underline{M}} = (X^{\underline{m}}, \Theta^{\underline{\mu}})$, and introduce the supersymmetric cotangent basis 1-forms in target space

$$\begin{array}{rcl} \Pi^{\underline{m}} & = & dX^{\underline{m}} - \tfrac{i}{2} d\Theta\Gamma^{\underline{m}}\Theta \ , \\ \Xi^{\underline{\mu}} & = & d\Theta^{\underline{\mu}} \ . \end{array} \tag{4.1}$$

An arbitrary frame is obtained by SO(1,6) rotations

$$\begin{array}{rcccl} E^{\underline{a}} & = & \Pi^{\underline{m}} u_{\underline{m}}{}^{\underline{a}} & = & dZ^{\underline{M}} E_{\underline{M}}{}^{\underline{a}} \ , \\ E^{\underline{\alpha}} & = & \Xi^{\underline{\mu}} u_{\underline{\mu}}{}^{\underline{\alpha}} & = & dZ^{\underline{M}} E_{\underline{M}}{}^{\underline{\alpha}} \ . \end{array} \tag{4.2}$$

Here $u_{\underline{m}}{}^{\underline{a}}$ and $u_{\underline{\mu}}{}^{\underline{\alpha}}$ are the "Lorentz harmonics". The embedding matrix $\mathscr{E}_A{}^{\underline{A}}$ is defined as the pullback of the target space 1-form $E^{\underline{A}}$ onto the world-volume:

$$\mathscr{E}_A{}^{\underline{A}} := E_A(f^* E^{\underline{A}}) = E_A{}^M(\partial_M Z^{\underline{M}}) E_{\underline{M}}{}^{\underline{A}} = (\nabla_A Z^{\underline{M}}) E_{\underline{M}}{}^{\underline{A}} \ . \tag{4.3}$$

Here ∇_A is the induced covariant derivative on the world-volume. The essential ingredient of the doubly supersymmetric approach is the "geometro-dynamical condition" [7,8], or the embedding condition [9]

$$\mathscr{E}_\alpha{}^{\underline{a}} = 0 \ . \tag{4.4}$$

Geometrically, this is simply the requirement that, at any point of $\mathscr{M}$, the odd tangent space to $\mathscr{M}$ lies entirely within the odd tangent space to $\underline{\mathscr{M}}$. In a number of interesting cases [9], the

[†] We do not strictly want to call these separate formalisms; rather we would like to reserve the former term for the specific procedure of extracting information about the dynamics from the torsion equation.

integrability condition for this constraint is so strong that it reproduces all the equations of motion for the extended object. This happens e.g. for the T5-brane in $D=11$ [9,4]. In the next section, however, we show that the embedding condition alone is not sufficient to put the $D=7$ 5-brane multiplet on-shell. It has to be augmented by a suitable constraint, as conjectured in ref. [4].

The embedding matrices can be read off from the induced vielbeins on the world volume. Expressed in terms of the Goldstone fields, they are, as mentioned in section 2:

$$\begin{array}{rcl} \mathscr{E}_a{}^{\underline{a}} & = & \delta_a{}^{\underline{a}} + i(\nabla_a\Phi)\delta_6{}^{\underline{a}} \; , \\ \mathscr{E}_\alpha{}^{\underline{\alpha}} & = & \delta_\alpha{}^{\underline{\alpha}} + (\nabla_\alpha\Theta^{\alpha'})\delta_{\alpha'}{}^{\underline{\alpha}} \; , \end{array} \tag{4.5}$$

and

$$\mathscr{E}_a{}^{\underline{\alpha}} = (\nabla_a\Theta^{\alpha'})\delta_{\alpha'}{}^{\underline{\alpha}} \; . \tag{4.6}$$

The embedding condition reads explicitly

$$\mathscr{E}_\alpha^{i\underline{a}} = \nabla_\alpha^i X^{\underline{a}} - \tfrac{i}{2}(\nabla_\alpha^i\Theta)\Gamma^{\underline{a}}\Theta = 0 \; . \tag{4.7}$$

In particular, $\mathscr{E}_\alpha^{i6}=0$ gives the $D=7$ non-linear "master constraint" of [9]:

$$\psi_\alpha^i = \nabla_\alpha^i\Phi \; , \tag{4.8}$$

(ψ being the normal spinor coordinate as in eq. (3.4)) as advertised in section 2. We know that the linearised version of the above constraint is not sufficient to put our theory on-shell. In the next section we show that this is also true at the non-linear level, without using a particular gauge, e.g. the static gauge.

Turning now to the induced world-volume torsion, it can be calculated from the integrability condition for the embedding matrix, $\nabla_{(\alpha}\mathscr{E}_{\beta)}{}^{\underline{a}}=0$, which gives

$$iT_{\alpha\beta}{}^c\mathscr{E}_c{}^{\underline{c}} = \mathscr{E}_\alpha{}^{\underline{\alpha}}\mathscr{E}_\beta{}^{\underline{\beta}}(\Gamma^{\underline{c}})_{\underline{\alpha\beta}} \; . \tag{4.9}$$

This is also known as the "twistor constraint" since $\mathscr{E}_\alpha{}^{\underline{\alpha}} = \nabla_\alpha^i\Theta^{\underline{\alpha}}$ is a twistor-like bosonic superfield. The world-volume torsion is then given by the equation

$$iT_{\alpha\beta}^{ij\,a} = \epsilon^{ij}(\gamma^a)_{\alpha\beta} + \epsilon_{kl}(\nabla_\alpha^i\nabla_\gamma^k\Phi)\gamma^a(\nabla_\beta^j\nabla_\delta^l\Phi) \tag{4.10}$$

(see eq. (2.45)), which is identical to the one obtained in the non-linear realisation formalism.

5. The equations of motion

We are now going to derive the equations of motion for the super-5-brane in 7 dimensions. As target space we will choose a flat $D=7$ superspace, i.e., all torsion components vanish except for

$$T_{\underline{\alpha\beta}}{}^{\underline{c}} = -i(\Gamma^{\underline{c}})_{\underline{\alpha\beta}} \ . \tag{5.1}$$

The intrinsic world-volume geometry is chosen to be $N=1$, $d=6$ conformal supergravity [17] and the constraints that we will need in order to obtain the equations of motion are

$$\mathscr{T}_{\alpha\beta}{}^{c} = -i(\Gamma^{c})_{\alpha\beta} \tag{5.2}$$

and

$$\mathscr{T}_{\alpha\beta}{}^{\gamma} = \mathscr{T}_{\alpha b}{}^{c} = \mathscr{T}_{ab}{}^{c} = 0 \ . \tag{5.3}$$

The fields occurring in the following equations are those found in the parametrisation (2.18) of the embedding matrix. We start by extracting the information hidden in (2.41) using the constraints (2.35) and (2.44). We thus obtain

$$\begin{array}{rrcl}
\text{(i)} & \hat{\bar{\mathscr{D}}}_{[a}m_{b]}{}^{c} & = & \frac{i}{2}(\chi_a\Gamma^c\chi_b) - L_{[ab]}{}^{d}m_d{}^{c} \ , \\
\text{(ii)} & m_{[a}{}^{d}K_{b]d}{}^{c'} & = & 0 \ , \\
\text{(iii)} & \mathscr{T}_{ab}{}^{\gamma} & = & 2\chi_{[a}{}^{\beta'}K_{b]\beta'}{}^{\gamma} \ , \\
\text{(iv)} & \hat{\bar{\mathscr{D}}}_{[a}\chi_{b]}{}^{\gamma'} & = & -\chi_{[a}{}^{\beta'}K_{b]\beta'}{}^{\gamma}h_{\gamma}{}^{\gamma'} - L_{[ab]}{}^{c}\chi_c{}^{\gamma'} \ , \\
\text{(v)} & \hat{\bar{\mathscr{D}}}_{\beta}m_a{}^{c} & = & i(\chi_a\Gamma^c h_\beta) - L_{\beta a}{}^{d}(m^{-1})_d{}^{c} \ , \\
\text{(vi)} & m_a{}^{d}K_{\beta d}{}^{c'} & = & i(\chi_a\Gamma^{c'})_\beta \ , \\
\text{(vii)} & \mathscr{T}_{a\beta}{}^{\gamma} & = & -\chi_a{}^{\beta'}K_{\beta\beta'}{}^{\gamma} - h_\beta{}^{\beta'}K_{a\beta'}{}^{\gamma} - L_{a\beta}{}^{\gamma} \ , \\
\text{(viii)} & \hat{\bar{\mathscr{D}}}_{a}h_\beta{}^{\gamma'} - \hat{\bar{\mathscr{D}}}_{\beta}\chi_a{}^{\gamma'} & = & L_{\beta a}{}^{d}\chi_d{}^{\gamma'} + h_\beta{}^{\beta'}K_{a\beta'}{}^{\gamma}h_\gamma{}^{\gamma'} \\
 & & & \quad +\chi_a{}^{\beta'}K_{\beta\beta'}{}^{\gamma}h_\gamma{}^{\gamma'} - K_{a\beta}{}^{\gamma'} \ , \\
\text{(ix)} & (\Gamma^d)_{\alpha\beta}m_d{}^{c} & = & (\Gamma^c)_{\alpha\beta} + (h_\alpha\Gamma^c h_\beta) \ , \\
\text{(x)} & 0 & = & h_{(\alpha}(\Gamma^{c'})_{\beta)} \ , \\
\text{(xi)} & L_{(\alpha\beta)}{}^{\gamma} + h_{(\beta}{}^{\beta'}K_{\alpha)\beta'}{}^{\gamma} & = & 0 \ , \\
\text{(xii)} & \hat{\bar{\mathscr{D}}}_{(\alpha}h_{\beta)}{}^{\gamma'} & = & \frac{i}{2}(\Gamma^c)_{\alpha\beta}\chi_c{}^{\gamma'} + h_{(\alpha}{}^{\beta'}K_{\beta)\beta'}{}^{\gamma}h_\gamma{}^{\gamma'} - K_{(\alpha\beta)}{}^{\gamma'} \ .
\end{array} \tag{5.4}$$

If we go through these equations we see that (i), (ii) and (iv) contain no information for the fields but simply describe parts of the torsion in the connection. Equations (iii) and (vii) determine the remaining world-volume torsion components in terms of the fields. Equation (v) does not generate any new fields and thus becomes an algebraic identity for the next-to-leading term in the superfield $m_a{}^b$. Two, more manifest, algebraic identities are (vi) and (xi). From (ix) and (x) we get

$$h_\alpha{}^{\beta'}=\tfrac{1}{6}(\Gamma^{abc})_\alpha{}^{\beta'}h_{abc} \tag{5.5}$$

and

$$m_a{}^b=\delta_a{}^b-2k_a{}^b\ , \tag{5.6}$$

where $k_a{}^b=h_{acd}h^{bcd}$. We note that putting $\mathscr{T}_{\alpha\beta}{}^c=-i(\Gamma^c)_{\alpha\beta}$ implies that

$$h^{(ij)}_{[\alpha\beta]}=0\ , \tag{5.7}$$

which is identical to the on-shell constraint imposed in the NR formalism of the previous sections.

In order to get the Dirac equation we take (xii):

$$\mathscr{K}_{(\alpha\beta)}{}^{\gamma'}=\tfrac{\mathrm{i}}{2}(\Gamma^c)_{\alpha\beta}\chi_c{}^{\gamma'} \tag{5.8}$$

and trace the three free spinor indices in different ways to extract the information. By applying $(\Gamma_d)^{\alpha\beta}$ and $(\Gamma^d)_{\gamma'}{}^\beta$ plus noting that

$$\mathscr{K}_\beta{}^{\gamma'}=\hat{\overline{\mathscr{D}}}h_\beta{}^{\gamma'}-\tfrac{1}{2}(\Gamma^b{}_{c'})_\beta{}^{\gamma'}\mathscr{K}_b{}^{c'} \tag{5.9}$$

we get

$$\chi_a{}^{\gamma'}=-\tfrac{\mathrm{i}}{4}(\Gamma_a)^{\alpha\beta}\mathscr{K}_{\alpha\beta}{}^{\gamma'} \tag{5.10}$$

and

$$i\Big(\chi_c{}^{\gamma'}(\Gamma^{cd})_{\gamma'\alpha}+\chi^d{}_\alpha\Big)=\tfrac{1}{2}(\Gamma^b{}_{a'}\Gamma^d)_\alpha{}^\beta\mathscr{K}_{\beta b}{}^{a'}-\tfrac{1}{6}(\Gamma^{abc}\Gamma^d)_\alpha{}^\beta\hat{\overline{\mathscr{D}}}_\beta h_{abc} \tag{5.11}$$

respectively. Now multiplying (5.11) by $(\Gamma_d)_{\delta'}{}^\alpha$, in order to get rid of h_{abc}, gives

$$(\Gamma^c)_{\delta'\gamma'}\chi_c{}^{\gamma'}=\tfrac{\mathrm{i}}{2}(\Gamma^b{}_{a'})_{\delta'}{}^\beta\mathscr{K}_{\beta b}{}^{a'}\ , \tag{5.12}$$

and if we use (5.10) in (5.12) we get

$$(\Gamma^d{}_{a'})_{\delta'}{}^\beta\mathscr{K}_{\beta d}{}^{a'}=0\ . \tag{5.13}$$

By comparing the two last equations we see that

$$(\Gamma^a)_{\alpha'\gamma'}\chi_a{}^{\gamma'} = 0 \ , \tag{5.14}$$

which is the Dirac equation.

In order to get the scalar and tensor equations of motion we take (viii):

$$\hat{\overline{\mathscr{D}}}_\beta \chi_a{}^{\gamma'} + Z_{a\beta}{}^{\gamma'} = \mathscr{K}_{a\beta}{}^{\gamma'} \ , \tag{5.15}$$

where

$$Z_{a\beta}{}^{\gamma'} := L_{\beta a}{}^d \chi_d{}^{\gamma'} + \chi_a{}^{\beta'} K_{\beta\beta'}{}^\gamma h_\gamma{}^{\gamma'} \ . \tag{5.16}$$

By using (5.9) in (5.15) we get

$$\overline{\mathscr{D}}_\beta \chi_a{}^{\gamma'} + Z_{a\beta}{}^{\gamma'} = (\tfrac{1}{6}\Gamma^{bcd}\hat{\overline{\mathscr{D}}}_a h_{bcd} - \tfrac{1}{2}\Gamma^b\Gamma_{c'}\mathscr{K}_{ab}{}^{c'})_\beta{}^{\gamma'} \ . \tag{5.17}$$

We now multiply (5.17) by $(\Gamma^{ae'})_{\gamma'}{}^\beta$ and use the Dirac equation, which gives us the scalar equation

$$\eta^{ab}\mathscr{K}_{ab}{}^{c'} = \tfrac{1}{4}(\Gamma^{ac'})_{\gamma'}{}^\beta Z_{a\beta}{}^{\gamma'} \ . \tag{5.18}$$

If we instead multiply (5.17) by $(\Gamma^a\Gamma_{ef})_{\gamma'}{}^\beta$ (and again use the Dirac equation) we get the tensor equation

$$\hat{\overline{\mathscr{D}}}{}^c h_{abc} = \tfrac{1}{8}(\Gamma^c\Gamma_{ab})_{\gamma'}{}^\beta Z_{c\beta}{}^{\gamma'} \ . \tag{5.19}$$

These equations of motion are analogous to the ones derived in ref. [4], and contain non-linearities of the same kind.

6. Summary and conclusions

We have given a detailed account of the geometry involved in embeddings of supermanifolds into supermanifolds. Special emphasis is put on the distinction between the different geometric objects encountered, since confusing e.g. intrinsic and induced geometry obscures the understanding of the formalism. Two preferred parametrisations of the embedding matrix in terms of matter fields, equations (2.7) and (2.18) have been presented, aiming towards distinct formulations of the world-volume field theory, each one emphasising different properties of the theory.

The second of these, referred to as the "embedding formalism", investigated by Howe, Sezgin and West [9], uses the torsion equation (2.41) together with the geometric "embedding condition" $\mathscr{E}_\alpha{}^{\underline{a}} = 0$ in order to derive equations of motion for the fields parametrising the embedding matrix (2.18). The second formulation occurs in the theory of non-linear realisations applied to the second supersymmetry (and the broken translations), as advocated by Bagger and Galperin [6]. By using the second parametrisation (2.7) of the embedding matrix, that formalism is rederived.

We also described briefly the transformations involved in going from one parametrisation to the other. Although it was straightforward to show that these transformations exist, we did not examine them in detail. As commented on in section 2, the transformations, eq. (2.10), represent a kind of local symmetry inherent in the definition of the embedding matrix, and it may be interesting to pursue the investigation further in order to extract information from the field redefinitions. We remind that the transformations involve not only the matter fields, but also the world-volume geometry.

We see two valuable aspects of this exercise. On one hand, the equivalence of two seemingly different starting points is established, and it becomes clear why they yield the same results (e.g. Born–Infeld dynamics). On the second hand it casts some light on the embedding procedure in explaining clearly why the obtained theory is one whose non-linearities stem from the (non-linearly realised) symmetry under the target space supersymmetry generators broken by the embedding.

The two parametrisations have been applied to a concrete case, namely the 5-brane of 7-dimensional supergravity. Here it was shown (in the first of the parametrisations) that the embedding condition alone did not provide enough information to put the theory on-shell. An additional irreducibility constraint, completely analogous to the one in the linear theory, had to be imposed. While this "algebraic" consideration became transparent in the language of non-linear realisations, the torsion components here become so complicated that we find the extraction of the field equations, though in principle possible, quite non-transparent. The second of the parametrisations, on the other hand, is quite suited for finding the field equations (section 5). In this case we did not need to impose any additional constraint after the world-volume torsion was chosen to be that of conformal 6-dimensional supergravity. Since we could associate the irreducibility constraint of our second formulation with the vanishing of a specific torsion component, we conjecture that the choice of torsion in the second case was more than a conventional one, so that the irreducibility constraint is hidden in the vanishing of the γ_{abc} part of the dimension-0 torsion component.

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Appendix A: Notation and conventions

Since a number of different geometric objects referring to different structures are encountered in this paper, we try to summarise the notation in the following table*.

	world-volume Intrinsic	world-volume Induced	Extrinsic	Normal	Target space
Metric	h	g		g'	$\underline{g}$
Vielbein	e_A	E_A		$E_{A'}$	$E_{\underline{A}}$
Connection	$\omega_A{}^B$	$\Omega_A{}^B$		$\Omega_{A'}{}^{B'}$	$\Omega_{\underline{A}}{}^{\underline{B}}$
Torsion	$\mathscr{T}^A$	T^A	$T_{AB}{}^{C'}$	$T^{A'}$	$T^{\underline{A}}$
Curvature	$\mathscr{R}_A{}^B$	$R_A{}^B$	$\mathscr{K}_{AB}{}^{C'}$	$R_{A'}{}^{B'}$	$R_{\underline{A}}{}^{\underline{B}}$
Exterior derivative	d	d		d'	$\underline{d}$
Canonical 1-form	θ	θ		θ'	$\underline{\theta}$
Covariant derivative	$\mathscr{D}$	∇		∇'	$\underline{\nabla}$

Appendix B: Spinors in 6 and 7 dimensions

The $D=7$ Γ-matrices decompose as

$$(\Gamma^a)_{\overline{\alpha\beta}} = \begin{pmatrix} (\Gamma^a)_{\alpha\beta} & 0 \\ 0 & (\bar{\Gamma}^a)_{\alpha'\beta'} \end{pmatrix} \tag{B.1}$$

and

$$(\Gamma^{a'})_{\overline{\alpha\beta}} = \begin{pmatrix} 0 & (\Gamma^{a'})_{\alpha\beta'} \\ (\Gamma^{a'})_{\alpha'\beta} & 0 \end{pmatrix} \tag{B.2}$$

with respect to the tangential and normal directions and they satisfy

$$(\Gamma^{\overline{a}})_{\underline{\alpha\beta}} = (\Gamma^{\overline{a}})_{\underline{\beta\alpha}} \ . \tag{B.3}$$

To raise and lower composite indices we use

$$C_{\overline{\alpha\beta}} = C^{\overline{\alpha\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \delta^{\alpha\beta'} \\ -\delta^{\alpha'\beta} & 0 \end{pmatrix} \ , \tag{B.4}$$

* Like the authors of ref. [7] we use the term "intrinsic" for the a priori defined world-volume entities, but note that there is an unfortunate disagreement on terminology. Mathematical literature may use the term for what we call "induced".

with the convention that

$$\begin{aligned} \psi^{\overline{\alpha}} &= C^{\overline{\alpha\beta}}\psi_{\overline{\beta}} \ , \\ \psi_{\overline{\beta}} &= \psi^{\overline{\beta}}C_{\overline{\beta\alpha}} \ . \end{aligned} \tag{B.5}$$

The algebra is

$$\{\Gamma^{\overline{a}},\Gamma^{\overline{b}}\} = 2\eta^{\overline{ab}} \ , \tag{B.6}$$

which implies that

$$\begin{aligned} \{\Gamma^a,\bar\Gamma^b\} :=&\Gamma^a\bar\Gamma^b+\Gamma^b\bar\Gamma^a = -2\eta^{ab}\delta_\alpha{}^\beta \ , \\ \{\bar\Gamma^a,\Gamma^b\} :=&\bar\Gamma^a\Gamma^b+\bar\Gamma^b\Gamma^a = -2\eta^{ab}\delta_{\alpha'}{}^{\beta'} \ , \\ \{\Gamma^{a'},\Gamma^{b'}\} :=&\Gamma^{a'}\Gamma^{b'}+\Gamma^{b'}\Gamma^{a'} = 2\eta^{a'b'}\delta_{\overline{\alpha}}{}^{\overline{\beta}} \ , \\ \{\Gamma^{a'},\Gamma^{b}\} :=&\Gamma^{a'}\Gamma^{b}+\Gamma^{b}\Gamma^{a'} = 0 \ , \end{aligned} \tag{B.7}$$

We split the 16 component indices according to

$$\begin{aligned} \psi_\alpha &\to \psi^i_\alpha \ , \\ \psi_{\alpha'} &\to \psi^\alpha_i \ , \end{aligned} \tag{B.8}$$

where after the split α is a Spin(1,5) index and i is a SU(2) index. For the Γ-matrices this implies

$$(\Gamma^a)_{\overline{\alpha}}{}^{\overline{\beta}} = \begin{pmatrix} 0 & -(\Gamma^a)_\alpha{}^{\beta'} \\ (\bar\Gamma^a)_{\alpha'}{}^\beta & 0 \end{pmatrix} \to \begin{pmatrix} 0 & -\varepsilon^{ij}(\gamma^a)_{\alpha\beta} \\ \varepsilon_{ij}(\bar\gamma^a)^{\alpha\beta} & 0 \end{pmatrix} \tag{B.9}$$

and

$$(\Gamma^{a'})_{\overline{\alpha}}{}^{\overline{\beta}} = \begin{pmatrix} (\Gamma^{a'})_\alpha{}^\beta & 0 \\ 0 & -(\Gamma^{a'})_{\alpha'}{}^{\beta'} \end{pmatrix} \to \begin{pmatrix} (\gamma^7)_\alpha{}^\beta\delta^i{}_j & 0 \\ 0 & -(\gamma^7)^\alpha{}_\beta\delta_i{}^j \end{pmatrix} \ , \tag{B.10}$$

where γ^a are the 6-dimensional gamma matrices [18]. They satisfy

$$\begin{aligned} (\gamma^a)_{\alpha\beta} &= -\,(\gamma^a)_{\beta\alpha} \ , \\ (\gamma_a)_{\alpha\beta}(\gamma^b)^{\alpha\beta} &= -\,4\delta_a{}^b \end{aligned} \tag{B.11}$$

and

$$(\gamma^a)_{\alpha\beta}(\gamma_a)_{\gamma\delta} = -2\varepsilon_{\alpha\beta\gamma\delta} \ . \tag{B.12}$$

Indices are raised and lowered according to

$$\begin{aligned} \psi^i =&\varepsilon^{ij}\psi_j \ , \\ \psi_i =&\psi^j\varepsilon_{ji} \end{aligned} \tag{B.13}$$

and

$$\begin{aligned}\psi^{\alpha\beta} &= \tfrac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}\psi_{\gamma\delta}\ ,\\ \psi_{\alpha\beta} &= \tfrac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}\psi^{\gamma\delta}\ .\end{aligned} \tag{B.14}$$

Notice that we can only raise and lower Spin(1,5) indices in pairs.

Appendix C: Some useful relations

In order to transform between vector and spinor indices we need the following relations, following from the lorentzian property of the u matrices:

$$\begin{aligned}(\overline{\mathscr{D}}u_{\overline{\alpha}}{}^{\underline{\gamma}})u_{\underline{\gamma}}{}^{\overline{\beta}} &= -\tfrac{1}{4}(\Gamma^{\overline{a}}{}_{\overline{b}})_{\overline{\alpha}}{}^{\overline{\beta}}(\overline{\mathscr{D}}u_{\overline{a}}{}^{\underline{c}})u_{\underline{c}}{}^{\overline{b}}\ ,\\ (\overline{\mathscr{D}}u_{\overline{a}}{}^{\underline{c}})u_{\underline{c}}{}^{\overline{b}} &= \tfrac{2}{\underline{n}}(\Gamma_{\overline{a}}{}^{\overline{b}})_{\overline{\beta}}{}^{\overline{\alpha}}(\overline{\mathscr{D}}u_{\overline{\alpha}}{}^{\underline{\gamma}})u_{\underline{\gamma}}{}^{\overline{\beta}}\ ,\end{aligned} \tag{C.1}$$

where $\underline{n}$ is the dimension of the target space spinor representation. If we take into account the split into tangential and normal indices we get

$$\begin{aligned}(\overline{\mathscr{D}}u_{\alpha}{}^{\underline{\gamma}})u_{\underline{\gamma}}{}^{\beta} &= -\tfrac{1}{4}\Big((\Gamma^{a}{}_{b})_{\alpha}{}^{\beta}(\overline{\mathscr{D}}u_{a}{}^{\underline{c}})u_{\underline{c}}{}^{b} + (\Gamma^{a'}{}_{b'})_{\alpha}{}^{\beta}(\overline{\mathscr{D}}u_{a'}{}^{\underline{c}})u_{\underline{c}}{}^{b'}\Big)\ ,\\ (\overline{\mathscr{D}}u_{\alpha}{}^{\underline{\gamma}})u_{\underline{\gamma}}{}^{\beta'} &= -\tfrac{1}{2}(\Gamma^{a}{}_{b'})_{\alpha}{}^{\beta'}(\overline{\mathscr{D}}u_{a}{}^{\underline{c}})u_{\underline{c}}{}^{b'} = -\tfrac{1}{2}(\Gamma^{a'}{}_{b})_{\alpha}{}^{\beta'}(\overline{\mathscr{D}}u_{a'}{}^{\underline{c}})u_{\underline{c}}{}^{b}\ ,\\ (\overline{\mathscr{D}}u_{\alpha'}{}^{\underline{\gamma}})u_{\underline{\gamma}}{}^{\beta'} &= -\tfrac{1}{4}\Big((\Gamma^{a}{}_{b})_{\alpha'}{}^{\beta'}(\overline{\mathscr{D}}u_{a}{}^{\underline{c}})u_{\underline{c}}{}^{b} + (\Gamma^{a'}{}_{b'})_{\alpha'}{}^{\beta'}(\overline{\mathscr{D}}u_{a'}{}^{\underline{c}})u_{\underline{c}}{}^{b'}\Big)\ ,\end{aligned} \tag{C.2}$$

and

$$\begin{aligned}(\overline{\mathscr{D}}u_{a}{}^{\underline{c}})u_{\underline{c}}{}^{b} &= \tfrac{4}{\underline{n}}(\Gamma_{a}{}^{b})_{\beta}{}^{\alpha}(\overline{\mathscr{D}}u_{\alpha}{}^{\underline{\gamma}})u_{\underline{\gamma}}{}^{\beta} = \tfrac{4}{\underline{n}}(\Gamma_{a}{}^{b})_{\beta'}{}^{\alpha'}(\overline{\mathscr{D}}u_{\alpha'}{}^{\underline{\gamma}})u_{\underline{\gamma}}{}^{\beta'}\ ,\\ (\overline{\mathscr{D}}u_{a'}{}^{\underline{c}})u_{\underline{c}}{}^{b'} &= \tfrac{4}{\underline{n}}(\Gamma_{a'}{}^{b'})_{\beta}{}^{\alpha}(\overline{\mathscr{D}}u_{\alpha}{}^{\underline{\gamma}})u_{\underline{\gamma}}{}^{\beta} = \tfrac{4}{\underline{n}}(\Gamma_{a'}{}^{b'})_{\beta'}{}^{\alpha'}(\overline{\mathscr{D}}u_{\alpha'}{}^{\underline{\gamma}})u_{\underline{\gamma}}{}^{\beta'}\ ,\\ (\overline{\mathscr{D}}u_{a}{}^{\underline{c}})u_{\underline{c}}{}^{b'} &= \tfrac{4}{\underline{n}}(\Gamma_{a}{}^{b'})_{\beta}{}^{\alpha'}(\overline{\mathscr{D}}u_{\alpha'}{}^{\underline{\gamma}})u_{\underline{\gamma}}{}^{\beta} = \tfrac{4}{\underline{n}}(\Gamma_{a}{}^{b'})_{\beta'}{}^{\alpha}(\overline{\mathscr{D}}u_{\alpha}{}^{\underline{\gamma}})u_{\underline{\gamma}}{}^{\beta'}\ .\end{aligned} \tag{C.3}$$

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Paper III

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Finite Tensor Deformations of Supergravity Solitons

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Abstract

We consider brane solutions where the tensor degrees of freedom are excited. Explicit solutions to the full non-linear supergravity equations of motion are given for the M5 and D3 branes, corresponding to finite selfdual tensor or Born–Infeld field strengths. The solutions are BPS-saturated and half-supersymmetric. The resulting metric space-times are analysed.

1. Introduction

The way in which branes in M theory and string theory arise as "soliton" solutions of 11- or 10-dimensional supergravity is well known, see *e.g.* [1,2]. Much less explored is the exact relation between the dynamics of the brane degrees of freedom and the target space fields. The former of course arise as zero-modes of the latter around a solitonic solution [3,4], but when one goes beyond a linear approximation, no such relation has been established so far. Part of the motivation of the present work is to fill this gap. Specifically, we address the question of finding solutions to the supergravity equations of motion corresponding to finite excitations of the tensorial degrees of freedom, while keeping the brane flat and infinite. The analysis is applied to the M5 brane of 11-dimensional supergravity and the D3 brane of type IIB supergravity, which both are truly solitonic. There are *a priori* strong reasons to believe that analytic solutions exist, since they are related to the dynamics of Born–Infeld vector fields and selfdual tensors on the world-volumes of the D3 and M5 branes, respectively. This calculation is carried through in section 2. Section 3 examines the metric properties of the resulting space-time, especially a limiting case for maximal field strength, where no asymptotic Minkowski region exists. In section 4, we show that the solutions are half-supersymmetric and construct the corresponding Killing spinors.

2. Finite tensor deformations

We want to find exact solutions for the M5 and D3 branes, where we have finite field strength deformations. What makes it possible to find exact solutions are the nice algebraic properties of the selfdual field strengths we are dealing with. For most of our conventions and notation we refer to ref. [4]. Here we just state our notation for the different types of indices occurring:
Space-time indices: $M, N, \ldots$ (coordinate-frame), $A, B, \ldots$ (inertial);
Longitudinal indices: $\mu, \nu, \ldots$ (coordinate-frame), $i, j, \ldots$ (inertial);
Transverse indices: $p, q, \ldots$ (coordinate-frame), $p', q', \ldots$ (inertial).

2.1. The M5 brane

The 4-form field strength H should be parametrised by a closed 3-form $F(x)$ lying in the longitudinal directions, according to experience from brane dynamics [5,6]. This can also be understood from the general Goldstone analysis [4]. In contrast to the (infinitesimal) Goldstone analysis, where F fulfilled a linear selfduality relation, F should in the exact analysis fulfill some non-linear selfduality relation. We are going to treat the simplest case where F is constant. Consider the equation of motion for H, $d{\star}H - \frac{1}{2}H{\wedge}H = 0$ [7]. The $\star$ operation involves the dualisation with the metric restricted to the 6-dimensional longitudinal directions. In the Goldstone analysis [4], where we considered an infinitesimal excitation of F, this metric was proportional to $1\!\!1$ and we did not have to care much about whether we had the radius-independent tensor in coordinate-frame or inertial indices, they just differed by a scalar function of the radial coordinate. Now, the dualisation in coordinate-frame indices involves a metric that will be "non-trivial", and for the selfduality to be consistent with radius-independence it must be possible to formulate it in terms of an inertial tensor.

Take h_{ijk} to be a (linearly) anti-selfdual inertial tensor. Define $q_{ij} = \frac{1}{2}h_i{}^{kl}h_{jkl}$. Then, $\mathrm{tr}\, q = 0$ and $q^2 = \mu 1\!\!1$, where $\mu = \frac{1}{6}\mathrm{tr}\, q^2$. The tensor $(qh)_{ijk} \equiv q_i{}^l h_{ljk}$ is automatically antisymmetric and selfdual. For later purposes, we define $\nu = \frac{1}{2}\sqrt{\mu}$. The most general Ansatz for the deformed 4-form is now

$$\begin{aligned} H_{\mu\nu\lambda p} &= e_\mu{}^i e_\nu{}^j e_\lambda{}^k \partial_p \Delta F_{ijk} \ , \\ F_{ijk} &= f h_{ijk} + g(qh)_{ijk} \ , \end{aligned} \tag{2.1}$$

where f and g are functions of μ and of the radial coordinate ρ. Due to the algebraic properties of h all higher order terms can be reduced to the two terms in the Ansatz. The necessity to include the second term is that the radial derivative on $\star H$ acts not only on the tensor but also on the vielbeins. The field along the 4-sphere will not change, since the magnetic charge should not be altered, so the background solution [8] remains unaltered

$$H_{pqrs} = \delta^{tu}\varepsilon_{pqrst}\partial_u \Delta \ , \tag{2.2}$$

where Δ is a harmonic function of the transverse coordinates, *i.e.*, $\delta^{pq}\partial_p\partial_q\Delta = 0$. By considering all functions, as f and g above, as functions of Δ instead of ρ, one covers AdS space ($\Delta = (\frac{R}{\rho})^3$) as well as the asymptotically flat brane solutions ($\Delta = 1 + (\frac{R}{\rho})^3$), without any extra calculational complication.

As an Ansatz for the vielbeins, we take

$$\begin{aligned} e_\mu{}^i &= \delta_\mu{}^j(a\delta_j{}^i + bq_j{}^i) \ , \\ e_p{}^{p'} &= c\delta_p{}^{p'} \ , \end{aligned} \tag{2.3}$$

where a, b and c are functions of μ and Δ. One thing that makes the calculations simpler is that all matrices that may occur, vielbeins and derivatives of vielbeins, commute with each other. One may quite easily calculate the Ricci tensor. A first observation is that the RHS of Einstein's equations can never contain $\partial_p\partial_q\Delta$, so such terms must not be present in R_{pq}. This implies that $c = (\det e_\mu{}^i)^{-1/\tilde{d}}$, where $\tilde{d} = D - d - 2$ ⋆. When this is used, the Ricci

⋆ D is the target space dimension and d that of the brane. Thus, in this case $\tilde{d}$=3.

tensor is, expressed in terms of $A \equiv \log e$ (e denoting $e_\mu{}^i$),

$$
\begin{aligned}
R_{pq} &= -\partial_p\Delta\partial_q\Delta\left(\mathrm{tr}(A'^2)+\tfrac{1}{d}(\mathrm{tr}A')^2\right)+\tfrac{1}{d}\delta_{pq}(\partial\Delta)^2\mathrm{tr}A''\ ,\\
R_{\mu\nu} &= -c^{-2}(\partial\Delta)^2 e_\mu{}^i e_\nu{}^j A''_{ij}\ .
\end{aligned}
\tag{2.4}
$$

Prime denotes differentiation w.r.t. Δ and $(\partial\Delta)^2 \equiv \delta^{pq}\partial_p\Delta\partial_q\Delta$. The matrix A will be parametrised as $A = \frac{1}{d}(\alpha 1\!\!1 + \beta q)$, and α is actually equal to $\log\det e_\mu{}^i$. It is also convenient to rescale the functions in the Ansatz for H as $\phi = e^{-\alpha}f$, $\psi = e^{-\alpha}g$. The remaining part of Einstein's equations, together with the e.o.m. for H, are now

$$
\begin{aligned}
0 &= \alpha'' - e^{2\alpha}(1-2\mu\phi\psi)\ ,\\
0 &= \beta'' + 3e^{2\alpha}(\phi^2+\mu\psi^2)\ ,\\
0 &= \alpha'^2 + \tfrac{1}{3}\mu\beta'^2 - e^{2\alpha}(1-4\mu\phi\psi)\ ,\\
0 &= \phi' + (e^\alpha + \tfrac{1}{2}\alpha')\phi - \tfrac{1}{2}\mu\beta'\psi\ ,\\
0 &= \psi' - (e^\alpha - \tfrac{1}{2}\alpha')\psi - \tfrac{1}{2}\beta'\phi\ .
\end{aligned}
\tag{2.5}
$$

This is one equation too many, but by differentiating the third equation one gets a combination of the other four (eventually, one has to check that the integration constant vanishes). The μ-dependence can be removed by redefining $\mu^{1/4}\phi \to \phi$, $\mu^{3/4}\psi \to \psi$, $\mu^{1/2}\beta \to \beta$; the equations become identical to the ones above with $\mu = 1$.

The background solution, describing either $\mathrm{AdS}_7 \times S^4$ or an M5 brane with no tensor excitations, is $\alpha = -\log\Delta$ and the rest zero. If one builds up the solution order by order in the perturbation, one first solves the zero-mode equation for ϕ giving $\phi = k\Delta^{-1/2}$. This linearised solution then backreacts on the geometry giving the lowest order perturbation to $\beta \sim \Delta^{-1}$. This non-diagonal metric then forces the tensor to contain the other duality component, $\psi \sim \Delta^{-3/2}$, which in turn enforces a diagonal modification to the vielbein, *i.e.* of α, of the order Δ^{-2}. And so it goes on. This becomes an expansion in negative powers of Δ and at the same time in the constant k, which just determines the normalisation of h_{ijk}. The μ-dependence is reinserted by choosing $\mu^{-1/4}k = 1$ (so that ϕ starts out with $\Delta^{-1/2}$), which makes the expansion look like

$$
\begin{aligned}
\alpha &\sim -\log\Delta + \mu\Delta^{-2} + \mu^2\Delta^{-4} + \ldots\\
\beta &\sim \Delta^{-1} + \mu\Delta^{-3} + \mu^2\Delta^{-5} + \ldots\\
\phi &\sim \Delta^{-1/2} + \mu\Delta^{-5/2} + \mu^2\Delta^{-9/2} + \ldots\\
\psi &\sim \Delta^{-3/2} + \mu\Delta^{-7/2} + \mu^2\Delta^{-11/2} + \ldots
\end{aligned}
\tag{2.6}
$$

Considering the first few terms in this expansion enabled us to find the exact solution:

$$\begin{aligned}\alpha &= -\frac{1}{2}\log(\Delta^2-\nu^2)\ ,\\ \beta &= \frac{3}{4\nu}\log\frac{\Delta-\nu}{\Delta+\nu}\ ,\\ \phi &= \frac{1}{2}\Big(\frac{1}{\sqrt{\Delta+\nu}}+\frac{1}{\sqrt{\Delta-\nu}}\Big)\ ,\\ \psi &= \frac{1}{4\nu}\Big(\frac{1}{\sqrt{\Delta+\nu}}-\frac{1}{\sqrt{\Delta-\nu}}\Big)\ .\end{aligned}\tag{2.7}$$

Before inserting the explicit solution for α and β in the metric, it is useful to note that the eigenvalues of the matrix q are $\pm 2\nu$, and that there are three of each. We group the longitudinal coordinates accordingly into $x_{\pm}$. The time direction is included in x_-. The metric then becomes

$$ds^2 = (\Delta^2-\nu^2)^{-1/6}\left[\left(\frac{\Delta+\nu}{\Delta-\nu}\right)^{1/2}dx_-^2+\left(\frac{\Delta-\nu}{\Delta+\nu}\right)^{1/2}dx_+^2\right]+(\Delta^2-\nu^2)^{1/3}dy^2\ .\tag{2.8}$$

It clearly reduces to the well known M5 brane metric when the tensor deformation is absent, *i.e.*, when $\nu=0$. We will return to the properties of the metric in section 3. Finally, inserting the solution into the Ansatz (2.1) gives us the 4-form in inertial indices:

$$H_{ijkp'} = \frac{\delta_{p'}{}^p\partial_p\Delta}{(\Delta^2-\nu^2)^{2/3}}\left[\frac{1}{\sqrt{\Delta+\nu}}\Pi_+h+\frac{1}{\sqrt{\Delta-\nu}}\Pi_-h\right]_{ijk}\ ,\tag{2.9}$$

where $\Pi_\pm=\frac{1}{2}(\mathbb{1}\pm\frac{q}{2\nu})$ project all indices on the $+$ and $-$ directions (the algebraic properties of h tell us that only $h_{i_+j_+k_+}$ and $h_{i_-j_-k_-}$ are non-vanishing).

2.2. The D3 brane

The relevant tensor field in type IIB supergravity [9] is the complex 3-form field strength H. The D3 brane is invariant under SL(2;$\mathbb{Z}$) transformations, and it is convenient to keep SL(2;$\mathbb{Z}$) covariance throughout the calculations. The Bianchi identity and equation of motion for H are

$$\begin{aligned}&DH-P\wedge\bar{H}=0\ ,\\ &D\star H-P\wedge\star\bar{H}+iG\wedge H=0\ ,\end{aligned}\tag{2.10}$$

where the U(1) covariant derivative D contains a connection Q, which together with P are the left-invariant SL(2;$\mathbb{R}$) Maurer–Cartan forms built from the scalars⋆.

⋆ We will leave Q out of the continued discussion—to the initiated reader it will be obvious that it is pure gauge, and we use this to put it to zero.

We use an Ansatz analogous to the M5 brane case:

$$H = d\Delta \wedge \tilde{F} \ , \tag{2.11}$$

where

$$\tilde{F}_{ij} = fF_{ij} + g\bar{F}_{ij} \ . \tag{2.12}$$

We again have one anti-selfdual ($\star F = -iF$) and one selfdual ($\star\bar{F} = i\bar{F}$) part. The algebraic properties of the matrix F are

$$\begin{aligned} &(FF)_{ij} = \mu\delta_{ij} \ ; \quad \mu = \tfrac{1}{4}\mathrm{tr}F^2 \ , \\ &(F\bar{F})_{ij} = (F\bar{F})_{ji} \ , \\ &\mathrm{tr}(F\bar{F}) = 0 \ . \end{aligned} \tag{2.13}$$

When we excite H we must also excite the 1-form P, as a consequence of the equations of motion, and we need an Ansatz for that too,

$$P = ud\Delta \ , \tag{2.14}$$

where $u = u(\mu, \bar{\mu}, \Delta)$. The Bianchi identity and equation of motion for P are

$$\begin{aligned} &DP = 0 \ , \\ &D\star P - H\wedge\star H = 0 \ . \end{aligned} \tag{2.15}$$

The Ansätze trivially fulfill the Bianchi identity parts of (2.10) and (2.15) when only functions of the radial coordinate are considered.

The Ansatz for the vielbeins are

$$\begin{aligned} &e_\mu{}^i = \delta_\mu{}^j(a\delta_j{}^i + b(F\bar{F})_j{}^i) \ , \\ &e_p{}^{p'} = c\delta_p{}^{p'} \ , \end{aligned} \tag{2.16}$$

which is also completely analogous to the M5 brane case, *i.e.*, $e_\mu{}^i$ is made up of the two symmetric matrices we can construct.

The Ricci tensor is given by eq. (2.4), where A is now parametrised as $A = \frac{1}{d}(\alpha 1\!\!1 + \beta F\bar{F})$. The equations of motion we want to solve are Einstein's equation

$$\begin{aligned} R_{MN} = 2\bar{P}_{(M}P_{N)} + \bar{H}_{(M}{}^{RS}H_{N)RS} - \tfrac{1}{12}g_{MN}\bar{H}_{RST}H^{RST} \\ + \tfrac{1}{96}G_{(M}{}^{RSTU}G_{N)RSTU} \ , \end{aligned} \tag{2.17}$$

together with the the equations of motion in (2.10) and (2.15). We use the background solution [10]

$$\begin{aligned} G = \pm\tfrac{1}{5!}(&\delta^{mn}\partial_m\Delta\varepsilon_{npqrst}dy^p\wedge dy^q\wedge dy^r\wedge dy^s\wedge dy^t \\ &-5g^{-2}\partial_m\Delta\varepsilon_{\mu\nu\rho\sigma}dy^m\wedge dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma)\ , \end{aligned} \tag{2.18}$$

where $g=\det(g_{MN})$ (the first term, which gives the D3 brane charge, is identical to the one in the ordinary D3 brane solution, and the second is its dual, where we have taken into account that the metric is modified). With the same rescalings as for the M5 brane, *i.e.*, $f=e^\alpha\phi$, $g=e^\alpha\psi$ and $u=e^\alpha\chi$, we can rewrite the equations of motion as

$$\begin{aligned} 0&=\alpha''-e^{2\alpha}\Big(1+4(\mu\phi\bar\psi+\bar\mu\bar\phi\psi)\Big)\ ,\\ 0&=\beta''-8e^{2\alpha}(\phi\bar\phi+\psi\bar\psi)\ ,\\ 0&=\alpha'^2+\tfrac{1}{2}\mu\bar\mu\beta'^2-e^{2\alpha}\Big(1+8(\mu\phi\bar\psi+\bar\mu\bar\phi\psi)-4\chi\bar\chi\Big)\ ,\\ 0&=\phi'+(e^\alpha+\tfrac{1}{2}\alpha')\phi-\tfrac{1}{2}\bar\mu\beta'\psi+e^\alpha\chi\bar\psi\ ,\\ 0&=\psi'-(e^\alpha-\tfrac{1}{2}\alpha')\psi-\tfrac{1}{2}\mu\beta'\phi+e^\alpha\chi\bar\phi\ ,\\ 0&=\chi'+\alpha'\chi+2e^\alpha(\mu\phi^2+\bar\mu\psi^2)\ . \end{aligned} \tag{2.19}$$

By differentiating the third equation we get a combination of the other five. The first three equations come from Einstein's equation, the fourth and fifth from the equation for H and the last one from the equation for P. From the properties of the fields involved under U(1) gauge transformations it is clear that α, β and ϕ are real functions, while ψ and χ must be real functions multiplied by μ. The solution to the equations is given by

$$\begin{aligned} \alpha&=-\frac{1}{2}\log(\Delta^2-\nu^2)\ ,\\ \beta&=-\frac{2}{\nu}\log\frac{\Delta-\nu}{\Delta+\nu}\ ,\\ \phi&=\frac{1}{2}\Big(\frac{1}{\sqrt{\Delta+\nu}}+\frac{1}{\sqrt{\Delta-\nu}}\Big)\ ,\\ \psi&=-\frac{\mu}{\nu}\Big(\frac{1}{\sqrt{\Delta+\nu}}-\frac{1}{\sqrt{\Delta-\nu}}\Big)\ ,\\ \chi&=\frac{\mu}{\sqrt{\Delta^2-\nu^2\ ,}} \end{aligned} \tag{2.20}$$

where $\nu=2|\mu|$ and we have used the normalisation that $\phi\to\Delta^{-1/2}$ as $\mu\to 0$ (the same rescaling argument holds here as for the M5 brane).

The metric may be diagonalised in the same manner as the M5 brane metric (the eigenvalues of $F\bar F$ are $\pm\frac{\nu}{2}$, and now time is in the positive eigenvalue sector), giving

$$ds^2=(\Delta^2-\nu^2)^{-1/4}\left[\left(\frac{\Delta+\nu}{\Delta-\nu}\right)^{1/2}dx_+^2+\left(\frac{\Delta-\nu}{\Delta+\nu}\right)^{1/2}dx_-^2\right]+(\Delta^2-\nu^2)^{1/4}dy^2\ . \tag{2.21}$$

Inserting the solution into the Ansätze (2.11) and (2.14) finally gives us

$$
\begin{aligned}
H_{p'ij} &= \frac{\delta_{p'}{}^{p}\partial_p\Delta}{2(\Delta^2-\nu^2)^{5/8}}\left[\frac{1}{\sqrt{\Delta+\nu}}(F-\frac{2\mu}{\nu}\bar{F})+\frac{1}{\sqrt{\Delta-\nu}}(F+\frac{2\mu}{\nu}\bar{F})\right]_{ij} ,\\
P_{p'} &= \frac{\mu\delta_{p'}{}^{p}\partial_p\Delta}{(\Delta^2-\nu^2)^{9/8}}
\end{aligned}
\qquad (2.22)
$$

in inertial indices.

We want to stress that the structures of the solutions for the D3 and M5 branes are completely analogous (except that we happen to excite additional scalar fields in the D3 brane case, which however is easily dealt with). The linear terms in the deformations, *i.e.*, the lowest order terms in the series expansions of ϕ, agree with the zero-modes derived in ref. [4].

3. Properties of the metrics

The metric space-times described by eqns. (2.8) and (2.21) represent deformations of the original AdS×sphere or brane space-times parametrised by one real number ν, measuring the square of the field strength. When the radial coordinate ρ runs from 0 (which is the horizon in the brane case and a subset of no special significance in the AdS case) to ∞, Δ runs from ∞ to 1 for the brane and from ∞ to zero for AdS. We see that there is potential danger when $\Delta-\nu$ becomes negative.

Let us first treat the AdS case. Here $\Delta-\nu=(\frac{R}{\rho})^{\tilde{d}}-\nu$, and this is bound to change sign at some finite radius when $\nu>0$. The question is whether this is a physical singularity or not. It is straightforward to calculate *e.g.* the curvature scalar, and find that it diverges at this radius. Such solutions do not define sensible space-times.

For the brane solutions, $\Delta-\nu=(\frac{R}{\rho})^{\tilde{d}}+1-\nu$. The solution makes sense for $\nu\leq 1$. This is a reflection of the Born–Infeld or Born–Infeld-like dynamics, which breaks down at field strengths where $\det(g+F)$ vanishes. The behaviour of the solutions for small radii is always unmodified, *i.e.*, $\mathrm{AdS}_{d+1}\times S^{\tilde{d}+1}$. For large radii, there is an asymptotic Minkowski region as long as ν is strictly smaller than 1.

The limiting case, $\nu=1$, has some interesting properties. One may calculate the curvature scalar, and find that it is non-singular as $\rho\to\infty$; it goes asymptotically as ρ^{-1}. After some trivial rescalings, the leading terms in the metric behave as

$$
\begin{aligned}
\text{M5:}\qquad ds^2 &= \rho^2dx_-^2+\rho^{-1}(dx_+^2+dy^2) ,\\
\text{D3:}\qquad ds^2 &= \rho^3dx_+^2+\rho^{-1}(dx_-^2+dy^2) .
\end{aligned}
\qquad (3.1)
$$

As $\rho \to \infty$, half of the longitudinal directions "expand" and the other half "shrink", and what remains is something rather like a continuously smeared membrane or string, respectively. Whether this interpretation is physically relevant is unclear to us, however it is supported by the asymptotic behaviour of the dual of the tensor field, which asymptotically lies in the shrinking directions and the $(\tilde{d}+1)$-sphere. The limiting metric does not factorise, but it has some things in common with the AdS metric: the space-like distance to $\rho = \infty$ is infinite, but light rays may reach infinity (and come back) in finite time.

4. Supersymmetric properties of the solutions

In the absence of an expectation value for the field strength on the brane, it is well known that the solutions break half the supersymmetry, *i.e.*, that there are 16 Killing spinors. Arguing naïvely in terms of the field theory on the brane, one might expect that giving a background value to F would break the entire remaining global supersymmetry, so that the solutions presented here would be non-supersymmetric (and perhaps less interesting). What actually happens is instead that there are new combinations of the broken and unbroken supersymmetries that become Killing spinors in the presence of $F \neq 0$, and that the new solutions enjoy the same amount of supersymmetry, 16 Killing spinors.

There are at least two good arguments why this should happen. The first, more conceptual, is that the tensor modes are very much on the same footing as the scalar ones, in the sense that they all result from breaking of large gauge transformations [4]. Deforming a brane by giving constant "field strength" to scalars (transverse coordinates) corresponds to tilting the brane through some angle, a somewhat trivial operation that of course does not change the number of supersymmetries. The definition of world-volume chirality however changes, and one has to recombine broken and unbroken supersymmetries to recover the new Killing spinors. A similar phenomenon should occur for the tensors, and we already know that an analogous mechanism is at work for the tensors themselves, where chirality (selfduality) becomes nonlinear. The second, more technical, argument is that one knows from work on κ-symmetry in supersymmetric brane dynamics [11,5,12,6] that there is a half-rank projection matrix, or generalised chirality operator [13], acting on spinors separating broken and unbroken supersymmetry, and that this matrix generically depends on F. For constant F, this means that there should be 16 global supersymmetries.

When the tensor degrees of freedom are turned on, the branes carry not only magnetic charge, but also local electric charge [14,4]. The BPS property expressed through the existence of a local projection on the Killing spinors involves both charges, which explains why the excited brane may be BPS-saturated although the tensor excitations carry energy. The configurations carrying global electric charge are world-volume solitons [15].

The most convincing argument is of course to construct the Killing spinors explicitly, which we now proceed to do (although we satisfy ourselves with the M5 brane case). The preserved supersymmetry obeys the Killing spinor equation obtained by setting the variation of the gravitino field in the background to zero:

$$\delta_\zeta\psi_M = D_M\zeta - \tfrac{1}{288}(\Gamma_M{}^{NPQR} - 8\delta_M{}^N\Gamma^{PQR})\zeta H_{NPQR} = 0 \ . \qquad (4.1)$$

The inertial gamma matrices are split as $\Gamma_A = (\gamma_i, \gamma_7 \oplus \Sigma_{p'})$ The calculation is straightforward (along the lines of ref. [4]). After assuming that the only functional dependence comes through Δ, one obtains a differential equation for ζ,

$$\zeta' + \left[\tfrac{1}{3}\Delta(\Delta^2-\nu^2)^{-1} + \tfrac{1}{4}(\Delta^2-\nu^2)^{-1/2}\gamma_7\right]\zeta = 0 \ , \qquad (4.2)$$

and an algebraic condition

$$\tfrac{1}{2}(1\!\!1 + \Gamma)\zeta = 0\,; \quad \Gamma = \Delta^{-1}(\Delta^2-\nu^2)^{1/2}\big(\gamma_7 + \tfrac{1}{12}(\Delta^2-\nu^2)^{1/2}F_{ijk}\gamma^{ijk}\big) \ . \qquad (4.3)$$

It is now crucial that the last equation projects ζ on half the original number of components. Using the explicit forms of the functions entering into F gives $\Gamma^2 = 1\!\!1$, so that eq. (4.3) is a projection. It defines a generalised chirality condition, which for any fixed radius takes the form known from the κ-symmetric formulation of the M5 brane [6]. The chirality condition varies continuously with the radial coordinate, as does the non-linear selfduality condition on F.

The solution to eq. (4.2) is

$$\begin{aligned} \zeta_- &= (\Delta^2-\nu^2)^{1/12}\left(\frac{1}{\sqrt{\Delta+\nu}} + \frac{1}{\sqrt{\Delta-\nu}}\right)^{1/2}\lambda_- \ , \\ \zeta_+ &= (\Delta^2-\nu^2)^{-5/12}\left(\frac{1}{\sqrt{\Delta+\nu}} + \frac{1}{\sqrt{\Delta-\nu}}\right)^{-1/2}\lambda_+ \ , \end{aligned} \qquad (4.4)$$

where ζ has been split in chirality components according to the eigenvalue of γ_7 and where $\lambda_\pm$ do not depend on Δ. We notice that in the absence of a tensor field we recover the Killing spinors of ref. [4] which was $\zeta = \Delta^{-1/12}\lambda_-$. It remains to be checked that the solutions (4.4) are consistent with the chirality (4.3), *i.e.*, that the Δ-dependence cancels upon inserting the solutions into the chirality condition. This indeed happens, and the chirality condition condenses into

$$\lambda_+ = -\tfrac{1}{12}h_{ijk}\gamma^{ijk}\lambda_- \ , \qquad (4.5)$$

which together with eq. (4.4) gives the explicit form of the Killing spinors.

5. Discussion

We have derived a new class of half-supersymmetric solutions of 11-dimensional and type IIB supergravity, corresponding to M5 and D3 branes with non-vanishing constant field strength. The structure of the solutions clearly reflects the property of Born–Infeld-like dynamics as opposed to quadratic actions, in that there is a maximal allowed value of the field strength.

It is interesting to note that although the symmetry of the solutions is smaller than in the case of vanishing field strength—the longitudinal SO(1,5) part of the isometry group is broken into SO(1,2)×SO(3) for the M5 brane (and accordingly for the D3 brane), the amount of supersymmetry is unchanged (the longitudinal translations of course remain unbroken). The split of the longitudinal directions in two groups is a novel property of brane solutions. It is not related to the longitudinal symmetry breaking induced by world-volume solitons, rather this split seems to have something to do with other branes, in these cases membranes and strings. The phenomenon might deserve further study, especially in the strong field limit. The formalism of ref. [16] may be useful in this context.

It should be possible to push the analysis further by considering also configurations with field strengths that depend on the longitudinal coordinates and thus derive the dynamics of the fields (the result would be in the selfdual form of refs. [12,6]). Another application would be the generalisation to other types of branes—the method presented here might provide a manifestly SL(2;$\mathbb{Z}$)-covariant formulation of the type IIB 5-branes. Finally, it would be interesting to understand whether the limiting solutions of maximal field strength have some physical significance, considering their interesting asymptotic structure.

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Paper IV

New insights in brane and Kaluza–Klein theory through almost product structures

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Abstract

We will show that gauge theory can be described by an almost product structure, which is a certain type of endomorphism of the tangent bundle. We will recover the gauge field strength as the Nijenhuis tensor of this endomorphism. We discuss a generalization to the case of a general Kaluza-Klein theory. Furthermore, we will look at the classification of these almost product structures in the case where we have a manifold with metric, and fit the M-brane solutions into this classification scheme. In this analysis certain algebraic properties of the space of differential forms and multivectors are obtained. All analysis is global but we will give local expressions where we find it suitable.

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1 Introduction

In this paper we will look at brane theory, gauge and Kaluza–Klein theory from a new perspective. The basic idea is that instead of looking at embedded branes or gauge theory over a base manifold, we will treat the total space directly. As is well known, this is how Kaluza–Klein theory works. There, the total manifold is constrained to have one part with a certain isometry group, that becomes the gauge group upon compactification.

Here we will generalize this analysis, in a completely global treatment, and show that Kaluza–Klein theory or normal gauge theory are nothing but special cases of almost product manifolds. The characterization of Kaluza–Klein theory will be that we have one foliation which is geodisable (*i.e.*, the almost product structure becomes a sort of Ehresmann connection), which serves as the fiber, and one perhaps non-integrable distribution. These split the tangent bundle of the total space into two different parts, in fiber bundle language called the vertical and horizontal respectively. The base space of a principal bundle is recovered as the leaf space of the foliation, and the field strength as the Nijenhuis tensor of the almost product structure. From this case of a geodisable foliation we will also find that imposing integrability on the normal distribution, the vanishing of the Nijenhuis tensor gives us two new coboundary operators under which the entire graded algebra of differential forms will become doubly graded. This gives us directly a topological splitting of the manifold into two parts, see also [5], why the cohomology groups split under these two coboundary operators.

In brane theory we discuss how the solutions in fact may be regarded as foliations of the total space rather than as embedded objects. In the case of *e.g.* the M5-branes we know that the solutions are non-singular [11, 10] but even in the cases where the solutions are singular it is clear that although the objects may be introduced as sources by Dirac delta functions, consistency of the theory in the total space demands that we cut out these points or sections of the manifold. This again brings us back to foliated space solutions. In M-brane theory we will see that the discussed solutions are indeed doubly foliated—the Nijenhuis tensor of the almost product structure, which will characterize the solution, vanishes. In the case of brane solutions, though, we are also interested in the metric and we will thus put the metric solution into the classification regime of almost product manifolds. We will also see that the solutions are characterized by certain basic forms, the anti-symmetric tensor fields, which can be seen to be compatible with the almost product structure characterized by the brane solution or equivalently as defining this almost product structure. In this new formalism we are also able to argue for the existence of new solutions to M-theory in which we only require the brane to be integrable and the foliation needs not even be Ehresmann. In these cases, ref. [9] argues that in situations like this, when the foliation does not define a fibration of the manifold, we could very well be up to a leaf space that is non-commutative.

The paper is divided into five section, of which you are now reading the first. When studying endomorphisms of the tangent bundle, which is the base for these almost product structures that will be dealt with, we find a certain dual structure on the set of differential forms and the set of multivectors on a manifold. In section two we will describe this dual picture and show how the set of derivations on differential forms and multivectors are recovered in a very easy and similar way. These relations will then be of utmost interest for us as we turn to the case of endomorphisms on the tangent bundle later in section four. But first we will review the basic concepts of distributions and foliations on a manifold expressed in a global way. This occurs in section three where we also get acquainted with the concept of basic and semi-basic forms. These are the keys of putting the anti-symmetric tensor fields into the context of foliations and therefore the brane solutions. We will also see that the set of basic forms are closed under the exterior derivative and we thus get the cohomology groups of the leafspace. This treatment in the first subsection of section three will be done completely without the presence of a metric. In the following subsection, though, we will see what additional structure the presence of a metric will give us in terms of distributions and foliations. Here we introduce the deformation tensor and look at the interpretation of its irreducible parts.

Section four is the core of the paper. Here we start in the first subsection to introduce endomorphisms on the tangent bundle in a general framework. We introduce the I-bracket associated with an endomorphism and the Nijenhuis tensor, measuring how far this endomorphism is from being a Lie algebra homomorphism on the infinite-dimensional Lie algebra of vector fields on a manifold. We will also treat the case where a metric is present, in which we introduce the Jordan bracket associated with an endomorphism and the Jordan tensor, measuring how much the Jordan bracket fails to commute with the endomorphism. Finally, we introduce a generalized deformation tensor, which later will reduce to the deformation tensor introduced in section three. In subsection two we will start to see how certain endomorphisms, namely almost product structures, will serve as characterizing possible foliations on a manifold. We will here recover the tensors from section three, where we see that the Nijenhuis tensor measures non-integrability and the Jordan tensor measures how far the two complementary distributions, defined by the almost product structure, are from being geodesic. We will introduce two new connections which both commute with the almost product structure and which will be of certain interest in the classification scheme in following subsection. In the last subsection, we present the classification and examine certain important consequences of some of its special cases. We will see the splitting of the cohomology groups in the case where the almost product structure defines two Ehresmann foliations, we will see how the holonomy groups split when the almost product structure is covariantly constant, we will see how the brane solutions fit into this classification scheme and we will see that the Nijenhuis tensor indeed measures the field strength in gauge theories and Kaluza–Klein theories. We will also present the local structures of the involved objects in some selected cases.

In section five we will end by discussing how this new formalism can help us in understanding M-theory, and how one could proceed in studying these objects further.

2 Derivations on the exterior algebra of forms and vectors

In this section we will take a look at the set of all derivations on both the set of differential forms and the set of multivectors on a manifold. We will find two different types of brackets, namely the Schouten–Nijenhuis bracket, which is a bracket between multivectors, and the Frölicher–Nijenhuis bracket, which is a bracket between vector-valued forms. The Schouten–Nijenhuis bracket will by the adjoint mapping become a derivation on the set of multivectors, while the Frölicher–Nijenhuis bracket turns up in the commutator of two derivations on the set of differential forms. See [19, 20] for a more detailed study. The new thing here is that we will put the action on forms and multivectors on an equal footing. We will see that all maps have its dual in the co-picture. But let us start with some preliminaries.

Definition 2.1
Let $\mathscr{M}$ be a manifold and let us denote

$$\begin{aligned} \Omega^p :=& \Omega^p(\mathscr{M}) && \textit{the } p\textit{-forms on } \mathscr{M} \\ \Omega :=& \bigoplus_p \Omega^p && \textit{the graded algebra of } p\textit{-forms} \\ \Lambda^q :=& L^q(\mathscr{M}) && \textit{the } q\textit{-vectors on } \mathscr{M} \\ \Lambda :=& \bigoplus_q \Lambda^q && \textit{the graded algebra of } q\textit{-vectors} \end{aligned}$$

We will now study maps on Ω and Λ, especially those maps that are derivations. So we need some basic definitions.

Definition 2.2
Let $D \in \mathrm{Lin}(\Omega, \Omega)$, $(D \in \mathrm{Lin}(\Lambda, \Lambda))$ be a linear map on the graded algebra of pforms (q-vectors). Then D is said to be graded of degree k if

$$D : \Omega^p \longmapsto \Omega^{p+k}, \quad (\Lambda^p \longmapsto \Lambda^{p+k}).$$

Let $D_i \in \mathrm{Lin}(\Omega, \Omega)$, $(D_i \in \mathrm{Lin}(\Lambda, \Lambda))$ be graded linear maps of degree k_i, then we can define the graded commutator by

$$[D_1, D_2] := D_1 \circ D_2 - (-1)^{k_1 k_2} D_2 \circ D_1,$$

which is again a graded linear map but of degree $k_1 + k_2$. We can also define the graded Jacobi bracket by

$$\begin{aligned} [D_1, D_2, D_3] :=& [[D_1, D_2], D_3] + (-1)^{k_1(k_2+k_3)}[[D_2, D_3], D_1] + \\ &+ (-1)^{k_3(k_1+k_2)}[[D_3, D_1], D_2] \end{aligned}$$

A graded linear map is said to be a graded derivation of degree k if

$$D(\omega_1 \wedge \omega_2) = D\omega_1 \wedge \omega_2 + (-1)^{kl} \omega_1 \wedge D\omega_2, \qquad \text{for } \omega_1 \in \Omega^l, (\Lambda^l),\ \omega_2 \in \Omega, (\Lambda).$$

We will denote the space of all derivations of degree k by $\mathrm{Der}_k\Omega$, $(\mathrm{Der}_k\Lambda)$ and the space of all derivations by

$$\mathrm{Der}\,\Omega := \bigoplus_k \mathrm{Der}_k\Omega, \quad (\mathrm{Der}\,\Lambda := \bigoplus_k \mathrm{Der}_k\Lambda).$$

Now it is easily seen that the set of all derivations on Ω and Λ respectively, forms a graded Lie algebra under the graded commutator.

Proposition 2.3
Der Ω (Der Λ) *becomes a graded Lie algebra with the graded commutator defined in 2.2. This means that it satisfies the graded Jacobi identity, i.e.,*

$$[D_1, D_2, D_3] = 0, \quad \forall D_i \in \mathrm{Der}\,\Omega,\ (\forall D_i \in \mathrm{Der}\,\Lambda).$$

proof: By direct calculation. ■

We will study these sets of derivations in the next two subsections. We will see that we can introduce a map called the generalized Lie derivative which is not necessarily a derivation but has some nice characteristics. Among these are the natural fact that it reduces to the usual Lie derivative in the case it acts by a vector, and the fact that it has a dual map. All this will become clear in subsection 2.

2.1 Derivations on Λ

Here we will see how we can obtain the Schouten–Nijenhuis bracket of two multivectors from the generalized Lie derivative to be introduced. We will start by introducing a formal boundary operator on the set of multivectors on $\mathscr{M}$ denoted by Λ.

Definition 2.4
Let Λ be the graded algebra of all p-vectors on $\mathscr{M}$. Then we can formally form a differential complex over the vector fields with the sequence

$$0 \xleftarrow{\ \partial\ } \Lambda^1 \xleftarrow{\ \partial\ } \Lambda^2 \cdots \xleftarrow{\ \partial\ } \Lambda^q \xleftarrow{\ \partial\ } \cdots$$

where ∂ is a "boundary" operator with the characteristics

$$\partial : \Lambda \longmapsto \Lambda, \quad \Lambda^q \longmapsto \Lambda^{q-1}, \quad \partial \circ \partial = 0.$$

It is defined on a p-vector by

$$\partial(X_1 \wedge \ldots \wedge X_p) := \sum_{i<j} (-1)^{i+j+1} [X_i, X_j] \wedge X_1 \wedge \overset{i}{\vee} \overset{j}{\vee} \wedge X_p,$$

where $\overset{i}{\vee}\ \overset{j}{\vee}$ means that X_i and X_j are omitted, and satisfies $\partial \Lambda^1 = 0$. The nilpotency follows from the Jacobi identity of the vector bracket.

Remark 2.5
It should be pointed out that ∂ defined above is no derivation. It is not defined on functions and not even well defined on general p-vectors.

We refer to the definition of ∂ as formal because of what we learned from remark 2.5. We can see why it is not well defined on p-vectors by taking the 2-vector example. Let $X_1 \wedge X_2 \in \Lambda^2$ be a 2-vector on $\mathscr{M}$, then we know that as a 2-vector $X_1 \wedge X_2 = f X_1 \wedge f^{-1} X_2$, where f is an arbitrary function on $\mathscr{M}$. But $\partial(X_1 \wedge X_2) = [X_1, X_2] \neq \partial(f X_1 \wedge f^{-1} X_2) = [X_1, X_2] + f^{-1} X_1[f] X_2 - f X_2[f^{-1}] X_1$, so we see that it is not well defined. It should be noted however that it is well defined on the set of multivectors on a Lie algebra where the functions instead becomes pure numbers. Nevertheless we will see that the formal boundary operator is of importance.

We also need to introduce the exterior product between multivectors.

Definition 2.6
Let Λ be the graded algebra of all p-vectors on $\mathscr{M}$, let $X \in \Lambda^1$ be a vector, then the **exterior product** *with respect to X have the following characteristics:*

$$\varepsilon_X : \Lambda \longmapsto \Lambda, \quad \Lambda^q \longmapsto \Lambda^{q+1}, \quad \varepsilon_X \circ \varepsilon_X = 0.$$

The exterior product is defined by its action on a p-vector by

$$\varepsilon_X(X_1 \wedge \ldots \wedge X_p) := X \wedge X_1 \wedge \ldots \wedge X_p.$$

Let $Y \in \Lambda^1$ be another vector, then

$$[\varepsilon_X, \varepsilon_Y] = 0$$

i.e., $\varepsilon_X \varepsilon_Y = -\varepsilon_Y \varepsilon_X$. Now let $X_i \in \Lambda^1$ be p vectors and let us extend the exterior product in the sense

$$\varepsilon_{X_1 \wedge \ldots \wedge X_p} := \varepsilon_{X_1} \circ \ldots \circ \varepsilon_{X_p}.$$

This makes the exterior product a p-graded map, i.e.,

$$\varepsilon_{X_1 \wedge \ldots \wedge X_p} : \Lambda \longmapsto \Lambda, \quad \Lambda^q \longmapsto \Lambda^{q+p}.$$

Remark 2.7
It should be noted that the map ε_X, although a linear map, is no derivation.

We can now create the generalized Lie derivative by taking the commutator of these two maps on Λ.

Definition 2.8
Let $X \in \Lambda$ be a p-vector on $\mathscr{M}$. We can then define the generalized Lie derivative, $\check{\mathscr{L}}_X$, with following characteristics:

$$\check{\mathscr{L}}_X : \Lambda \longmapsto \Lambda, \quad \Lambda^q \longmapsto \Lambda^{q+p-1}.$$

It is defined simply through the boundary operator and the exterior product by

$$\check{\mathscr{L}}_X := [\partial, \varepsilon_X].$$

Remark 2.9
The generalized Lie derivative, $\check{\mathscr{L}}_X$ is only a derivation in the case when $X \in \Lambda^1$ is a vector. In this case it is of course the usual Lie derivative.

So to sum up we have three maps on Λ, the boundary operator ∂, the exterior product ε_X and the generalized Lie derivative $\check{\mathscr{L}}_X$, of whom neither is a derivation except for the case when X is a pure vector when the generalized Lie derivative reduces to the ordinary Lie derivative. Now, however, if we take the commutator of two generalized Lie derivatives we recover the Schouten–Nijenhuis bracket which as the adjoint mapping is a derivation on Λ.

Definition 2.10
Let $X \in \Lambda^p$, $Y \in \Lambda^q$ be two multivectors on $\mathscr{M}$ and let $\omega \in \Omega^{p+q-1}$ a closed $(p+q-1)$-form, then we can define the **Schouten–Nijenhuis bracket** *with following characteristics:*

$$[X,Y] : \Lambda \times \Lambda \longmapsto \Lambda, \quad \Lambda^p \times \Lambda^q \longmapsto \Lambda^{p+q-1},$$

or in the sense of adjoint mapping

$$\begin{aligned} \mathrm{ad}_X : \Lambda \longmapsto \Lambda, \quad \Lambda^q \longmapsto \Lambda^{q+p-1} \\ \mathrm{ad}_X Y := [X,Y]. \end{aligned}$$

The following definitions of the Schouten–Nijenhuis bracket are equivalent

$$\begin{aligned}
&(i) && \check{\mathscr{L}}_{[X,Y]} := [\check{\mathscr{L}}_X, \check{\mathscr{L}}_Y] \\
&(ii) && (-1)^{p-1}[X,Y] := \partial(X \wedge Y) - \partial X \wedge Y - (-1)^p X \wedge \partial Y \\
&(iii) && [X,Y] := \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \overset{i}{\cdots} \wedge X_p \wedge Y_1 \wedge \overset{j}{\cdots} \wedge Y_q
\end{aligned}$$

Proposition 2.11
The Schouten–Nijenhuis bracket as a map $\mathrm{ad}_X : \Lambda \mapsto \Lambda$ is a derivation on Λ and satisfies the graded Jacobi identity. It forms thus a Lie algebra structure on $\mathrm{Der}\,\Lambda$.

We will see in next subsection, where we look at derivations on the set of differential forms Ω on $\mathscr{M}$, that the boundary operator will become a kind of dual to the exterior derivative or co-boundary operator on Ω, the exterior product will be dual to the interior product and the generalized Lie derivative will become dual to a generalized Lie derivative on differential forms.

2.2 The dual maps on Λ and Ω

We will here define the exterior derivative, the interior product and the generalized Lie derivatives as dual maps to those defined in the previous subsection.

Definition 2.12
Let Ω be the graded algebra of all p-forms on $\mathscr{M}$ and let us form the differential complex over Ω with sequence

$$0 \xrightarrow{\ i_*\ } \Omega^0 \xrightarrow{\ d\ } \Omega^1 \xrightarrow{\ d\ } \cdots \Omega^p \xrightarrow{\ d\ } \cdots$$

where i_* is an inclusion, and d is the coboundary operator on Ω with following characteristics:

$$d : \Omega \longmapsto \Omega, \quad \Omega^p \longmapsto \Omega^{p+1}, \quad d \circ d = 0.$$

Let $X_i \in \Lambda^1$ be vector fields on $\mathscr{M}$ then we can define the coboundary operator by

$$\begin{aligned} d\omega(X_1, \ldots, X_{p+1}) \quad := \quad & \sum_i (-1)^{i+1} \mathscr{L}_{X_i} \omega(X_1, \overset{i}{\check{\ldots}}, X_{p+1}) - \\ & -\omega(\partial(X_1 \wedge \ldots \wedge X_{p+1})) \end{aligned}$$

.

The nilpotency is not manifest but follows by the relation $[\mathscr{L}_{X_i}, \mathscr{L}_{X_j}] = \mathscr{L}_{[X_i, X_j]}$. We see that the coboundary operator in some sense is the adjoint operator of ∂.

We see that although the boundary operator defined in 2.4 was not well defined, the total expression for the exterior derivative is. The exterior derivative is of course a derivation, hence its name. Now to the interior product.

Definition 2.13
Let Ω be the graded algebra of p-forms on $\mathscr{M}$, $X \in \Lambda^1$ be a vector field and let ε_X be the exterior product defined in 2.6. Then we can define the interior product with the following characteristics:

$$i_X : \Omega \longmapsto \Omega, \quad \Omega^p \longmapsto \Omega^{p-1}, \quad i_X \circ i_X = 0,$$

as the adjoint of the exterior product, i.e.,

$$i_X \omega(X_1, \ldots, X_{p-1}) = \omega(\varepsilon_X(X_1 \wedge \ldots \wedge X_{p-1})).$$

The interior product satisfies $i_X \Omega^0 = 0$ and

$$[i_X, i_Y] = 0,$$

where $Y \in \Lambda^1$. We can in fact extend the interior product in the same way as we extended the exterior product so that for $X_i \in \Lambda^1$

$$i_{X_1 \wedge \ldots \wedge X_q} := i_{X_1} \circ \ldots \circ i_{X_q}$$

satisfies

$$i_{X_1 \wedge \ldots \wedge X_q} \omega(X_{q+1}, \ldots, X_p) = \omega(\varepsilon_{X_1 \wedge \ldots \wedge X_q} X_{q+1} \wedge \ldots \wedge X_p)$$

and is therefore a q graded map, i.e.,

$$i_{X_1 \wedge \ldots \wedge X_q} : \Omega \longmapsto \Omega, \quad \Omega^p \longmapsto \Omega^{p-q}.$$

Remark 2.14
The interior product, i_X, is a derivation only when $X \in \Lambda^1$ is a vector field.

If we recall the definition of the ordinary Lie derivative acting on forms we immediately see how this can be generalized.

Definition 2.15
Let Ω be the graded algebra of p-forms on $\mathscr{M}$, let $X \in \Lambda^1$ be a vector field. Let $\mathscr{L}_X$ be the Lie derivative with following characteristics:

$$\mathscr{L}_X : \Omega \longmapsto \Omega, \quad \Omega^p \longmapsto \Omega^p.$$

It is defined by

$$(\mathscr{L}_X\omega)(X_1, \ldots, X_p) = \mathscr{L}_X\omega(X_1, \ldots, X_p) - \omega(\mathscr{L}_X(X_1 \wedge \ldots \wedge X_p))$$

and satisfies Cartan's infinitesimal homotopy formula

$$\mathscr{L}_X = [i_X, d].$$

Let $Y \in \Lambda^1$ be another vector field. The Lie derivative satisfies the following equations:

$$\begin{array}{rcl} [\mathscr{L}_X, d] & = & 0 \\ [\mathscr{L}_X, i_Y] & = & i_{[X,Y]} \\ [\mathscr{L}_X, \mathscr{L}_Y] & = & \mathscr{L}_{[X,Y]} \end{array}$$

So we can proceed, as in the previous subsection, by introducing the generalized Lie derivative acting on forms by simply generalizing Cartan's formula.

Definition 2.16
Let $X \in \Lambda^q$ be a multivector on $\mathscr{M}$, then the generalized Lie derivative on p-forms is a map with following characteristics:

$$\hat{\mathscr{L}}_X : \Omega \longmapsto \Omega, \quad \Omega^p \longmapsto \Omega^{p-q+1},$$

and is defined by

$$\hat{\mathscr{L}}_X := [i_X, d].$$

Remark 2.17
This generalized Lie derivative, $\hat{\mathscr{L}}_X$, acting on forms is only a derivation on Ω when X is a vector field.

By this remark we have a similar case to that of the previous subsection, now however we do not know for sure that this map is dual to the generalized Lie derivative acting on multivectors introduced before, but we have to show this.

Proposition 2.18
Let $X = X_1 \wedge \ldots \wedge X_q \in \Lambda^q$ be a q-vector and let $\omega \in \Omega^p$ be a p-form then the generalized Lie derivative satisfies

$$\begin{aligned}(\hat{\mathscr{L}}_X\omega)(X_{q+1}, \ldots, X_{p+1}) = & \sum_{i=1}^{q} (-1)^{i+1} \mathscr{L}_{X_i}\omega(X_1, \overset{i}{\ldots\vee\ldots}, X_q, X_{q+1}, \ldots, X_{p+1}) - \\ & - \omega(\check{\mathscr{L}}_X(X_q \wedge \ldots \wedge X_{p+1}))\end{aligned}$$

proof: By direct calculation:

$$
\begin{aligned}
(\hat{\mathscr{L}}_X\omega)(X_{q+1},\dots,X_{p+1}) = {}& (i_{X_1\wedge\ldots\wedge X_q}d - (-1)^q d i_{X_1\wedge\ldots\wedge X_q})(\omega)(X_{q+1},\dots,X_{p+1}) = \\
={}& d\omega(\varepsilon_{X_1\wedge\ldots\wedge X_q}X_{q+1}\wedge\ldots\wedge X_{p+1}) - \\
& (-1)^q\sum_{i=q+1}^{p+1}(-1)^{i-q+1}\mathscr{L}_{X_i}(i_{X_1\wedge\ldots\wedge X_q}\omega)(X_{q+1},\overset{i}{\check{\ldots}},X_{p+1}) + \\
& (-1)^q(i_{X_1\wedge\ldots\wedge X_q}\omega)(\partial(X_{q+1}\wedge\ldots\wedge X_{p+1})) = \\
={}& \sum_{i=1}^{p+1}(-1)^{i+1}\mathscr{L}_{X_i}\omega(X_1,\overset{i}{\check{\ldots}},X_{p+1}) - \omega(\partial\varepsilon_{X_1\wedge\ldots\wedge X_q}X_{q+1}\wedge\ldots\wedge X_{p+1}) - \\
& -\sum_{i=q+1}^{p+1}(-1)^{i+1}\mathscr{L}_{X_i}\omega(X_1,\dots,X_q,X_{q+1},\overset{i}{\check{\ldots}},X_{p+1}) + \\
& (-1)^q\omega(\varepsilon_{X_1\wedge\ldots\wedge X_q}\partial X_{q+1}\wedge\ldots\wedge X_{p+1}) = \\
={}& \sum_{i=1}^{q}(-1)^{i+1}\mathscr{L}_{X_i}\omega(X_1,\overset{i}{\check{\ldots}},X_q,X_{q+1},\dots,X_{p+1}) - \\
& -\omega(\check{\mathscr{L}}_X(X_q\wedge\ldots\wedge X_{p+1}))
\end{aligned}
$$

■

Put together we have now seen that all these maps come with their duals. This is pointed out in following remark.

Remark 2.19
We see that these operators are formally adjoints to each others as acting on forms and multivectors respectively and we can write

$$
\begin{array}{rcl}
\Omega & & \Lambda \\
d & \longleftrightarrow & \partial \\
i_X & \longleftrightarrow & \varepsilon_X \\
\hat{\mathscr{L}}_X := [i_X, d] & \longleftrightarrow & [\partial, \varepsilon_X] =: \check{\mathscr{L}}_X
\end{array}
$$

as a correspondence table.

Now as we saw that we recovered the Schouten–Nijenhuis bracket when taking the commutator of two generalized Lie derivatives acting on multivectors we shall find out that we will get the same thing for the generalized Lie derivative acting on forms (up to a sign).

Proposition 2.20
Let $X \in \Lambda^p$ and $Y \in \Lambda^q$ be two multivectors. Then the brackets defined through the generalized Lie derivatives, i.e.,

$$
\begin{aligned}
\check{\mathscr{L}}_{[X,Y]^{\vee}} &:= [\check{\mathscr{L}}_X, \check{\mathscr{L}}_Y] \\
\hat{\mathscr{L}}_{[X,Y]^{\wedge}} &:= [\hat{\mathscr{L}}_X, \check{\mathscr{L}}_Y]
\end{aligned}
$$

are related by

$$[X,Y]^{\wedge} = -[Y,X]^{\vee} = (-1)^{(p-1)(q-1)}[X,Y]^{\vee}$$

proof: By combinatorics. ■

2.3 Derivations on Ω

In this subsection we will look at the set of all derivations on the set of differential forms Ω on $\mathscr{M}$. We will see that they are spanned by mappings involving vector valued forms denoted Ω_1^p, but as before we will start by looking at these mappings acting on Λ and then see that their duals acting on forms are derivations. So lets first start with the exterior product.

Definition 2.21
Let $I \in \Omega_1^p$ be a vector-valued p-form on $\mathscr{M}$, then the exterior product ε_I of I is a map with following characteristics:

$$\varepsilon_I : \Lambda \longmapsto \Lambda, \quad \Lambda^q \longmapsto \Lambda^{q-p+1},$$

and if $\mu \in \mathrm{Perm}(p+q)$ we can define the exterior product of I on a $(p+q)$-vector by

$$\varepsilon_I(X_1 \wedge \ldots \wedge X_{p+q}) := \frac{1}{p!q!} \sum_{\mu} (-1)^{\mu} I(X_{\mu_1}, \ldots, X_{\mu_p}) \wedge \ldots \wedge X_{\mu_{p+q}}$$

Remark 2.22
If $I \in \Omega_1^1$ is a endomorphism, i.e., a 1-1 tensor, then ε_I is a derivation on Λ.

We can now define the generalized Lie derivative of a vector-valued form acting on differential forms by the immediate analogue of definition 2.16.

Definition 2.23
Let $I \in \Omega_1^p$ be a vector-valued p-form on $\mathscr{M}$, then let us define the generalized Lie derivative acting as a map on Λ with following characteristics:

$$\check{\mathscr{L}}_I : \Lambda \longmapsto \Lambda, \quad \Lambda^q \longmapsto \Lambda^{q-p}$$

We define it in analogous way as before by

$$\check{\mathscr{L}}_I := [\partial, \varepsilon_I]$$

From the definition above we can now find the expression for the generalized Lie derivative in terms of the ordinary commutator on vectors.

Proposition 2.24
Let $I \in \Omega_1^p$ be a vector-valued p-form on $\mathscr{M}$, let $X_i \in \Lambda^1$ be vector fields and $\mu \in \mathrm{Perm}(p+q)$. Then

$$\check{\mathscr{L}}_I(X_1 \wedge \ldots \wedge X_{p+q}) = \frac{1}{p!(q-1)!} \sum_{\mu} (-1)^{\mu} [I(X_{\mu_1}, \ldots, X_{\mu_p}), X_{\mu_{p+1}}] \wedge \ldots \wedge X_{\mu_{p+q}} -$$
$$- \frac{(-1)^{p-1}}{(p-1)!(q-1)!2!} \sum_{\mu} (-1)^{\mu} I([X_{\mu_1}, X_{\mu_2}], \ldots, X_{\mu_{p+1}}) \wedge \ldots \wedge X_{\mu_{p+q}}$$

proof:

[19] plus the proof of 2.29 ■

Now we are ready to study the set of derivations on Ω. We will not go into details but only stress the differences appearing with this dual picture and refer to [19] for a more detailed study. To start with we will define what we mean by a algebraic derivation.

Definition 2.25
Let $D \in \operatorname{Der}\Omega$ then D is said to be algebraic if

$$D|\Omega^0 = 0$$

Let $\omega \in \Omega$ be a p-form on $\mathscr{M}$, then if D is **algebraic** *we have*

$$D(f\omega) = fD\omega, \quad \forall f \in C^\infty(\mathscr{M})$$

which means that D is tensorial.

We will see that the set of algebraic derivations on Ω is spanned by the interior product of vector valued forms on $\mathscr{M}$, so let us define the interior product again as the dual map to the exterior product.

Definition 2.26
Let $I \in \Omega_1^p$ be a vector-valued p-form on $\mathscr{M}$ and let $\omega \in \Omega^q$ be a q-form. Then define the interior product of I on Ω as a map with following characteristics:

$$i_I : \Omega \longmapsto \Omega, \quad \Omega^q \longmapsto \Omega^{q+p-1}.$$

Let $\omega \in \Omega^q$ be a q-form and define the internal product as the formal adjoint to the exterior product as

$$i_I\omega(X_1, \dots , X_{p+q-1}) := \omega(\varepsilon_I(X_1 \wedge \dots \wedge X_{p+q-1})).$$

Now [19] tells us that not only is the map i_I an algebraic derivation on Ω, but that all algebraic derivations can be written in that way, so we have a one-to-one correspondence.

Proposition 2.27
Let $D \in \operatorname{Der}_k \Omega$ be a graded derivation of degree k, then

$$D = i_I$$

for some $I \in \Omega_1^{k+1}$.

proof: See [19]. ■

It is also clear that if we again introduce the generalized Lie derivative of vector-valued forms by the analogue to Cartan's formula we know that it must be a derivation because it is now a commutator of two derivations.

Definition 2.28
Let $I \in \Omega_1^p$ be a vector-valued p-form on $\mathscr{M}$ and define the generalized Lie derivative on q-forms as a map with following characteristics:

$$\hat{\mathscr{L}}_I : \Omega \longmapsto \Omega, \quad \Omega^q \longmapsto \Omega^{q+p},$$

defined by

$$\hat{\mathscr{L}}_I := [i_I, d]$$

What is not clear is that it again is dual to the generalized Lie derivative acting on multivectors defined in 2.23 which indeed is no derivation on Λ unless I is a vector.

Proposition 2.29
Let $I \in \Omega_1^p$ be a vector-valued p-form on $\mathscr{M}$, $\omega \in \Omega^q$ a q-form and $X_i \in \Lambda^1$ be vectors. Then

$$\begin{aligned}(\hat{\mathscr{L}}_I\omega)(X_1, \dots, X_{p+q}) =& \frac{1}{p!q!}\sum_{\mu}(-1)^{\mu}\mathscr{L}_{I(X_{\mu_1},\dots,X_{\mu_p})}\omega(X_{\mu_{p+1}}, \dots, X_{\mu_{p+q}}) - \\ &- \omega(\check{\mathscr{L}}_I(X_1 \wedge \dots \wedge X_{p+q}))\end{aligned}$$

proof: The proof is by direct calculation,

$$\begin{aligned}(\hat{\mathscr{L}}_I\omega)&(X_1, \dots, X_{p+q}) = ((i_I d - (-1)^{p-1} d i_I)\omega)(X_1, \dots, X_{p+q}) = \\ =& d\omega(\varepsilon_I(X_1 \wedge \dots \wedge X_{p+q})) \\ &- (-1)^{p-1}\left(\sum_i (-1)^{i+1}\mathscr{L}_{X_i}(i_I\omega)(X_1, \overset{i}{\dots\checkmark}, X_{p+q}) - i_I\omega(\partial(X_1 \wedge \dots \wedge X_{p+q}))\right) = \\ =& \frac{1}{p!q!}\sum_{\mu}(-1)^{\mu} d\omega(I(X_{\mu_1}, \dots, X_{\mu_p}) \wedge \dots \wedge X_{\mu_{p+q}}) \\ &- (-1)^{p-1}\left(\sum_i (-1)^{i+1}\mathscr{L}_{X_i}(i_I\omega)(X_1, \overset{i}{\dots\checkmark}, X_{p+q}) - \omega(\varepsilon_I\partial(X_1 \wedge \dots \wedge X_{p+q}))\right) = \\ =& \frac{1}{p!q!}\sum_{\mu}(-1)^{\mu}\Big(\mathscr{L}_{I(X_{\mu_1},\dots,X_{\mu_p})}\omega(X_{\mu_{p+1}} \wedge \dots \wedge X_{\mu_{p+q}}) + \\ &+ q(-1)^{p-1}(-1)^{\mu_i}\mathscr{L}_{X_{\mu_i}}\omega(I(X_{\mu_1}, \dots, X_{\mu_p}) \wedge \overset{\mu_i}{\dots\checkmark} \wedge X_{\mu_{p+q}})\Big) \\ &- \omega(\partial\varepsilon_I(X_1 \wedge \dots \wedge X_{p+q})) \\ &- (-1)^{p-1}\left(\sum_i (-1)^{i+1}\mathscr{L}_{X_i}\omega(\varepsilon_I(X_1, \overset{i}{\dots\checkmark}, X_{p+q})) - \omega(\varepsilon_I\partial(X_1 \wedge \dots \wedge X_{p+q}))\right) = \\ =& \frac{1}{p!q!}\sum_{\mu}(-1)^{\mu}\mathscr{L}_{I(X_{\mu_1},\dots,X_{\mu_p})}\omega(X_{\mu_{p+1}}, \dots, X_{\mu_{p+q}}) - \\ &- \omega(\check{\mathscr{L}}_I(X_1 \wedge \dots \wedge X_{p+q}))\end{aligned}$$

∎

From [19] we also know that any derivation can be split into two parts, one part which is algebraic and one which looks like the generalized Lie derivative.

Proposition 2.30
Let $D \in \mathrm{Der}_k\Omega$ be a derivation of degree K on $\mathscr{M}$ then it can be uniquely be decomposed like

$$D = \hat{\mathscr{L}}_I + i_J$$

for some $I \in \Omega_1^k$, $J \in \Omega_1^{k+1}$. Furthermore we have the following equivalences

$$\begin{aligned} I = 0 &\iff D \text{ algebraic} \\ J = 0 &\iff [D, d] = 0 \end{aligned}$$

proof: See [19]. ■

Again we can introduce a bracket by looking at the commutator of two generalized Lie derivations. This bracket is the Frölicher–Nijenhuis bracket.

Definition 2.31
Let $I \in \Omega_1^p$, $J \in \Omega_1^q$ be two vector-valued forms on $\mathscr{M}$ and let Ω_1 denote the set of all vector-valued forms on $\mathscr{M}$ then we define the **Frölicher–Nijenhuis bracket** *with following characteristics:*

$$[I, J] : \Omega_1 \times \Omega_1 \longmapsto \Omega_1, \quad \Omega_1^p \times \Omega_1^q \longmapsto \Omega_1^{p+q}$$

Let $\mu \in \mathrm{Perm}(p+q)$, then the following definitions of the Frölicher–Nijenhuis bracket are equivalent

$$\begin{aligned} &(i) & \hat{\mathscr{L}}_{[I,J]} :=& [\hat{\mathscr{L}}_I, \hat{\mathscr{L}}_J] \\ &(ii) & [I, J](X_1, \ldots, X_{p+q}) :=& \frac{1}{p!q!} \sum_\mu (-1)^\mu [I(X_{\mu_1}, \ldots, X_{\mu_p}), J(X_{\mu_{p+1}}, \ldots, X_{\mu_{p+q}})] + \\ &&& - J(\check{\mathscr{L}}_I(X_1 \wedge \ldots \wedge X_{p+q})) + (-1)^{pq} I(\check{\mathscr{L}}_J(X_1 \wedge \ldots \wedge X_{p+q})) \end{aligned}$$

proof: [19] plus 2.24 ■

We will also find that if we define the bracket by the commutator of two generalized Lie derivations acting on multivectors we will again get the Frölicher–Nijenhuis bracket up to a sign.

Proposition 2.32
Let $I \in \Omega_1^p$ and $J \in \Omega_1^q$ be two vector-valued forms then the brackets defined through the generalized Lie derivatives, i.e.,

$$\begin{aligned} \check{\mathscr{L}}_{[I,J\check{]}} &:= [\check{\mathscr{L}}_I, \check{\mathscr{L}}_J] \\ \hat{\mathscr{L}}_{[I,J\hat{]}} &:= [\hat{\mathscr{L}}_I, \check{\mathscr{L}}_J] \end{aligned}$$

are related by

$$[I, J\hat{]} = -[J, I\check{]} = (-1)^{pq}[I, J\check{]}$$

proof: By combinatorics. ■

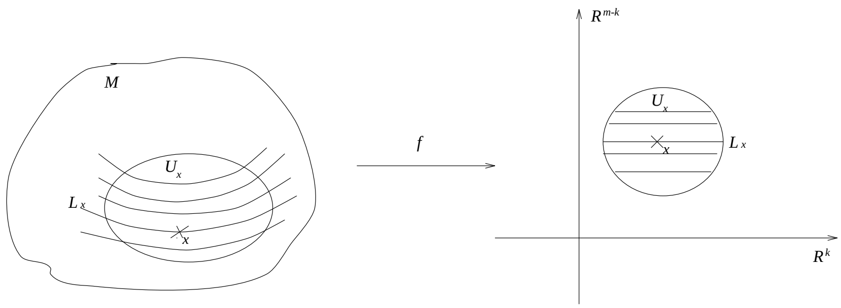

Figure 1: Foliation

3 Distributions and foliations

In this section we will review the basic concepts of distributions and foliations on a manifold. We will start with a general treatment in the first part, where neither a metric nor a connection is needed. We will see how we, from the solutions of the M2,5-branes, find that the anti-symmetric tensor field of the solution itself defines a foliation on the manifold. In the latter part we will see how we can give these concepts more structure by adding a metric and a connection.

3.1 General treatment

First we need to understand the basic concepts of distributions and foliations, so let us start by defining these.

Definition 3.1
Let $\mathscr{M}$ be a manifold with tangent bundle $T\mathscr{M}$, then a **distribution** *on $\mathscr{M}$ is a subset of the tangent bundle such that, for any point x in $\mathscr{M}$, the fiber $\mathcal{D}_x = \mathcal{D} \cap T_x\mathscr{M}$ is a vector subspace of $T_x\mathscr{M}$. The dimension of $\mathcal{D}_x$ is called the* **rank** *of the distribution. We will denote the distributions of constant rank k-distributions, where k is the rank.*

Definition 3.2
Let $\mathscr{M}$ be a manifold with dimension m, then a (k-) foliation, $\mathcal{F}$, is a family of connected subsets, $\mathcal{F} = \{\mathcal{L}_\alpha\}$, called leaves, such that

(i). $\underset{\alpha}{\cup}\mathcal{L}_\alpha = \mathscr{M}$

(ii). $\mathcal{L}_\alpha \cap \mathcal{L}_\beta = 0, \quad \alpha \neq \beta$

(iii). For any point $x \in \mathscr{M}$ there exists a local coordinate system (chart (U_x, φ))in which the leaves are coordinate surfaces.

It is clear from the definition of a foliation that it trivially defines a k-distribution and that this always locally can be spanned by coordinate vectors. If $\mathscr{M}$ is a manifold and $x^{\underline{m}}$ are local coordinates in a patch U then we will split it to $x^{\underline{m}} =$

$(x^m, y^{m'})$ where the leaves of the foliation are determined by local coordinate surfaces like $y^{m'} = C^{m'}$. The distribution associated with the leaves is then spanned by $\{\partial/\partial_m\}$ which are the annihilators of the normal pfaffian forms of the surfaces, $dy^{m'}$. We will see that this distribution is trivially integrable, but let us first define the concept.

Definition 3.3
Let $\mathcal{D}$ be a k-distribution on a manifold $\mathscr{M}$, then the set of all vectors in $\mathcal{D}$ forms a graded algebra on $\mathscr{M}$ with the usual wedge product. We will denote this algebra

$$\Lambda^q_{\mathcal{D}} := \Lambda^q(\mathscr{M})|_{\mathcal{D}}$$
$$\Lambda_{\mathcal{D}} := \bigoplus_q \Lambda^q_{\mathcal{D}}$$

where $\Lambda^q_{\mathcal{D}}$ is the set of q-vectors lying in $\wedge^q\mathcal{D}$. This algebra is a subalgebra of Λ, i.e., $\Lambda_{\mathcal{D}} \subset \Lambda$. The distribution, $\mathcal{D}$, is said to be **integrable** *if the algebra $\Lambda_{\mathcal{D}}$ is closed under the Schouten–Nijenhuis bracket, that is*

$$[\Lambda_{\mathcal{D}}, \Lambda_{\mathcal{D}}] \subset \Lambda_{\mathcal{D}}.$$

Remark 3.4
The usual definition of integrability of a distribution is that, taken any two vectors $X, Y \in \Lambda^1_{\mathcal{D}}$, the commutator of these vector fields will still be a vector field of the distribution, or

$$[X, Y] \subset \Lambda^1_{\mathcal{D}}, \quad \forall X, Y \in \Lambda^1_{\mathcal{D}},$$

but from the definition of the Schouten–Nijenhuis bracket we trivially see that the above definition of integrability is the same. The basic property of integrability is of course the existence of an integral manifold at every point, $x \in \mathscr{M}$. Integrability also assures that this integral manifold is unique and that the dimension is equal to the rank of the integrable distribution.

Now obviously the distribution associated with the leaves of the foliation is integrable because it is locally spanned by coordinate vectors and any two vectors built from these will be closed under the bracket in the sense that the resulting vector will again lie in the span of these coordinate vectors. Now one can go even further and prove that in fact any distribution of constant rank that is integrable also defines a foliation.

Proposition 3.5
Let $\mathcal{D}$ be a k-distribution on a manifold $\mathscr{M}$, then $\mathcal{D}$ defines a foliation if and only if $\mathcal{D}$ is integrable. Furthermore the leaves of this foliation are the integral manifolds of the distribution $\mathcal{D}$.

proof: See [8]. ■

So we get a 1-1 relation between the concept of an integrable distribution of constant rank and a foliation.

Remark 3.6
We see from the definition of a foliation and the equivalence to an integrable distribution that if $\mathcal{F}$ is an integrable distribution and $X \in \Lambda_{\mathcal{F}}$ is a vector field lying in the distribution then in every patch there exist coordinates $(x^m, y^{m'})$ such that the vector field X can be expressed locally as

$$X = X^m \partial_m = X^m(x, y) \frac{\partial}{\partial x^m}.$$

The coordinate surfaces $y^{m'} = C^{m'}$ are the leaves and $\partial/\partial x^m$ are the basis vectors along the leaves.

We have seen that a k-distribution can be imposed as a subset of the set of p-vectors on $\mathscr{M}$, which of course truncates at $k+1$, and that it in fact is a subalgebra under the Schouten–Nijenhuis bracket if and only if it defines a foliation. But now we want to see how we can understand this in the co-picture, where we look at the set of p-forms instead. So lets start with some basic definitions.

Definition 3.7
Let $\mathcal{D}$ be a k-distribution on a manifold $\mathscr{M}$, then the **annihilator** *or the* **codistribution** *of a distribution is denoted by, $\mathcal{D}^{*\prime}$, and defined by*

$$\mathcal{D}^{*\prime} := \bigcup_{x \in \mathscr{M}} \mathcal{D}^{*\prime}_x$$

where

$$\mathcal{D}^{*\prime}_x := \{\omega \in T^*_x\mathscr{M} : \ i_X \omega = 0, \ \forall X \in \Lambda^1_{\mathcal{D}}\}.$$

The set of all pfaffian forms in $\mathcal{D}^{*\prime}$ forms a graded algebra on $\mathscr{M}$ under the wedge product. The algebra is denoted by

$$\Omega^p_{\mathcal{D}^{*\prime}} := \Omega^p(\mathscr{M})|_{\mathcal{D}^{*\prime}},$$
$$\Omega_{\mathcal{D}^{*\prime}} := \bigoplus_p \Omega^p_{\mathcal{D}^{*\prime}}$$

where $\Omega^p_{\mathcal{D}^{*\prime}}$ is the set of p-forms lying in $\wedge^p \mathcal{D}^{*\prime}$. This algebra is a subset of the algebra of differential forms on $\mathscr{M}$, i.e., $\Omega_{\mathcal{D}^{*\prime}} \subset \Omega$.

Definition 3.8
Let $\mathcal{D}$ be a k-distribution on a manifold $\mathscr{M}$, then the ideal of $\mathcal{D}$ is defined by

$$I_{\mathcal{D}} := \bigoplus_p I^p_{\mathcal{D}},$$
$$I^p_{\mathcal{D}} = \{\omega \in \Omega^p : \omega(X_1, \dots, X_p) = 0, \ \forall X_i \in \Lambda^1_{\mathcal{D}}\},$$

then $I_{\mathcal{D}} \subset \Omega$ is to a subset of Ω.

Remark 3.9
We see that $\Omega_{\mathcal{D}^{*\prime}} \subset I_{\mathcal{D}}$, so the ideal of $\mathcal{D}$ is bigger than the set of forms spanned by the codistribution. We can picture these two types of forms in the case when

$\mathcal{D}$ is integrable and the annihilator locally is spanned by the pfaffian forms $\{dy^{m'}\}$, by

$$\begin{aligned}\omega =&\omega_{m'_1-m'_p}dy^{m'_1}\wedge\ldots\wedge dy^{m'_p}\\ \eta =&\eta_{m'_1-m'_p}\varphi^{m'_1}\wedge\ldots\wedge\varphi^{m'_{p-1}}\wedge dy^{m'_p}\end{aligned}$$

where $\omega\in\Omega^p_{\mathcal{D}^{*\prime}}$, $\eta\in I^p_{\mathcal{D}}$ and $\varphi^{m'}$ are arbitrary pfaffian forms.

The reason for introducing both these two types of subsets of the graded algebra of exterior forms on $\mathscr{M}$ is that, although the subset $\Omega_{\mathcal{D}^{*\prime}}$ seems more natural, we need the ideal to test the integrability of the distribution. In fact we have following proposition.

Proposition 3.10
Let $\mathcal{D}$ be a k-distribution on a manifold $\mathscr{M}$, let $I_{\mathcal{D}}$ be the ideal of $\mathcal{D}$ then $\mathcal{D}$ is integrable if and only if $I_{\mathcal{D}}$ is closed under the exterior derivative, i.e.,

$$dI_{\mathcal{D}}\subset I_{\mathcal{D}}.$$

proof: It is sufficient to prove that $d\omega\subset I^2_{\mathcal{D}}$ for all pfaffian forms in $\Omega_{\mathcal{D}^{*\prime}}$. So let $\omega\in\Omega^1_{\mathcal{D}^{*\prime}}$ and $X,Y\in\Lambda^1_{\mathcal{D}}$ then

$$d\omega(X,Y)=X[\omega(Y)]-Y[\omega(X)]-\omega([X,Y])=-\omega([X,Y])$$

which is zero for all vector fields in $\Lambda^1_{\mathcal{D}}$ if and only if the commutator lies in $\Lambda^1_{\mathcal{D}}$, *i.e.*, the distribution is integrable. ■

So we see that we can equivalently express the integrability of the distribution in the co-picture. Now we want to see the structure of the forms belonging to these subsets of forms, especially those belonging to $\Omega_{\mathcal{D}^{*\prime}}$. So we will make some preliminary definitions.

Definition 3.11
Let $\mathcal{D}$ be a k-distribution on a riemannian manifold $\mathscr{M}$, let $\omega\in\wedge T^*\mathscr{M}$ be a differential form on $\mathscr{M}$ and let $X\in\mathcal{D}$ be a vector field of the distribution, then we call ω

$$\begin{array}{rlll}(i) & \textbf{semi}-\textbf{basic},\ if & i_X\omega=0, & \\ (ii) & \textbf{invariant},\ if & & \mathscr{L}_X\omega=0,\\ (iii) & \textbf{basic},\ if & i_X\omega=0, & \mathscr{L}_X\omega=0,\end{array}$$

$\forall X\in\mathcal{D}$, with respect to $\mathcal{D}$. In the case when the form is basic it is also called an absolute integral invariant and equivalently satisfies $i_X\omega=0,\quad i_Xd\omega=0$.

We now see that the set of semi-basic forms in fact are those forms belonging to $\Omega_{\mathcal{D}^{*\prime}}$. But we also noted that they in general not are closed under the exterior derivative, not even in the case when $\mathcal{D}$ is integrable, but we had to introduce the ideal to express the integrability. Now the set of basic forms do indeed close under the exterior derivative. We can see the difference between semi-basic and basic forms in the case when the distribution is integrable in the following remark.

Remark 3.12
Let $\mathcal{F}$ be a foliation and $\Omega_{\mathcal{F}^{*\prime}}$ be graded algebra of the annihilator of $\mathcal{F}$, then $\Omega_{\mathcal{F}^{*\prime}}$ is the set of semi-basic forms with respect to $\mathcal{F}$. Let $\omega \in \Omega^p_{\mathcal{F}^{*\prime}}$ be a semi-basic p-form, it can then be expressed locally as

$$\omega = \omega_{m'_1 \cdots m'_p}(x, y) dy^{m'_1} \wedge \ldots \wedge dy^{m'_p}.$$

If additionally $\mathscr{L}_X \omega = 0,\ X \in \Lambda^1_{\mathcal{F}}$, then ω is basic and can then be expressed locally as

$$\omega = \omega_{m'_1 \cdots m'_p}(y) dy^{m'_1} \wedge \ldots \wedge dy^{m'_p}.$$

It should also be pointed out that if $X \in \Lambda_{\mathcal{F}}$ is a multivector on $\mathscr{M}$, tangent to the leaves, the basic forms are those forms vanishing under the generalized Lie derivative, i.e. $\hat{\mathscr{L}}_X \omega = 0, \quad \forall X \in \Lambda_{\mathcal{F}}$.

So we see that in the integrable case the basic forms are those that are semi-basic and constant along the leaves. As these forms are closed under the exterior derivative, we can look at cohomology groups on the leaf space.

Definition 3.13
The set of basic forms of a foliation $\mathcal{F}$ is a subset of $\Omega_{\mathcal{F}^{*\prime}}$ which we will denote $\Omega_{B_{\mathcal{F}}}$. The basic forms are closed under the exterior derivative, i.e.,

$$d : \Omega_{B_{\mathcal{F}}} \longmapsto \Omega_{B_{\mathcal{F}}}, \quad \Omega^p_{B_{\mathcal{F}}} \longmapsto \Omega^{p+1}_{B_{\mathcal{F}}}$$

so the basic forms form a subcomplex of the De Rahm complex. We can build the set of closed basic p-forms, $Z^p_{B_{\mathcal{F}}}$, and the set of exact basic p-forms, $B^p_{B_{\mathcal{F}}}$, and form the basic cohomology groups

$$\begin{aligned} H^p_{B_{\mathcal{F}}} &:= Z^p_{B_{\mathcal{F}}} / B^p_{B_{\mathcal{F}}}, \\ H_{B_{\mathcal{F}}} &:= \bigoplus_p H^p_{B_{\mathcal{F}}} \end{aligned}$$

which is the De Rahm cohomology of the leafspace of the foliation.

It shall be noted that although the manifold is nice the leaf space need not be. In fact [9] argues that in certain cases it is in fact non-commutative, and the basic cohomology groups can be infinite-dimensional even though $\mathscr{M}$ is compact. We will not discuss these basic cohomology groups here but refer to [24]. We can however say that it is easy to show that $H^1_{B_{\mathcal{F}}} \subset H^1(\mathscr{M})$.

We will now turn our study to the case when we are given a p-form and see what this specific p-form can tell us in the sense of distributions.

Definition 3.14
Let $\omega \in \Omega^p$ be a p-form on $\mathscr{M}$, then the **kernel** *of ω and the* **rank** *of ω at $x \in \mathscr{M}$, denoted $\ker_x \omega$ and $\mathrm{rank}_x \omega$ respectively, are defined through the kernel and the rank of the map $f_\omega|_x : \Lambda^1_x \mapsto \Omega^{p-1}_x$, defined by*

$$f_\omega(X)|_x := i_X \omega|_x, \quad X \in \Lambda^1.$$

Of course a p-form does not, in general, be of constant rank, but if it is we simply denote it by rank ω. The rank and the kernel is of course dual to each other in the sense that $\dim \ker_x \omega + \mathrm{rank}_x\omega = m$ where m is the dimension of the manifold, $\mathscr{M}$. Now we can, given a specific p-form, make the following definition.

Definition 3.15
Let $\omega \in \Omega$ be a differential form on $\mathscr{M}$, the **characteristic subspace**, *$\mathcal{D}_x$, of ω at a point $x \in \mathscr{M}$ is defined by*

$$\mathcal{D}_x := \ker_x \omega \cap \ker_x d\omega.$$

The **class** *of ω at x is the codimension of $\mathcal{D}_x$ in $T_x\mathscr{M}$ and the* **characteristic distribution**, *$\mathcal{D}$, of ω is simply $\mathcal{D} := \cup_{x\in\mathscr{M}}\mathcal{D}_x$.*

Remark 3.16
The class of a differential form is the smallest number of variables by which we can express it locally. If ω is a closed form then the class is equal to its rank.

To get a little better grip of what the class of a p-form is let us consider the four-dimensional Yang–Mills theory.

Example 3.17

> *Let F be the Lie algebra valued field strength of a abelian gauge potential A in a 4-dimensional space $\mathscr{M}_4$ then of course F is a Lie algebra valued two-form on $\mathscr{M}_4$ and by [6] we know that the rank of F is either 2 or 4. If the rank is two we know from definition 3.15 and the fact that F is closed that its characteristic distribution, which for an F of constant rank would be a characteristic foliation, would be two-dimensional. If this was the case we would for instance know that in a flat manifold we could choose coordinates in such a way that the two-dimensional foliation would be global coordinate surfaces. Now F would not depend on these coordinates but should effectively be a two-dimensional field strength. This is clearly not the case, and this is because the rank of F is in fact four and this is due to the self-duality condition $F = *F$.*

It should be noted that the set of p-forms of constant class is of great importance. The reason for this becomes clear by the following proposition.

Proposition 3.18
Let $\omega \in \Omega$ be a differential form on $\mathscr{M}$ with constant class, then the characteristic distribution will be of constant dimension and the distribution will be integrable, i.e., ω will define a foliation on $\mathscr{M}$.

proof: Take $X, Y \in \Lambda^1_{\mathcal{D}}$ where $\mathcal{D}$ is the characteristic distribution of ω, then $\mathscr{L}_{[X,Y]} = \mathscr{L}_X\mathscr{L}_Y\omega - \mathscr{L}_Y\mathscr{L}_X\omega = 0$, and $i_{[X,Y]}\omega = \mathscr{L}_X i_Y\omega - i_Y\mathscr{L}_X\omega = 0$ which implies $[X,Y] \in \Lambda^1_{\mathcal{D}}$, so $\mathcal{D}$ is integrable and thus a foliation. ■

Corollary 3.19
Let $\omega \in \Omega$ be a differential form on $\mathscr{M}$ with constant class, then ω is basic with respect to the characteristic foliation defined by ω.

proof: By definition. ■

So a p-form of constant class defines a foliation on $\mathscr{M}$ but then we know that finding a foliation on $\mathscr{M}$ of dimension k is equal to finding a p-form on $\mathscr{M}$ of constant class $m-k$. Note that the p-form need not be a $m-k$ form but can be an arbitrary p-form as long as $p \leq m-k$ and not a 0-form.

We look at the $M5$-brane solution.

Example 3.20

Let F_4 be the four-form in $D = 11$ supergravity and consider the M5-brane solution to the equations of motions in $\mathscr{M}_{11}$ [14, 10],

$$F = \partial^m \Delta(y) \varepsilon_{mnpqr} dy^n \wedge dy^p \wedge dy^q \wedge dy^r$$

with coordinates (x^μ, y^m) along the brane and transverse to the brane respectively. Then $\ker F = 6$ and the rank of F is $5 = 11 - 6$. Since F is closed we know that the class of F is equal to the rank and is therefore equal to 5. The characteristic distribution of F is nothing but the $M5$-brane itself which indeed is integrable and thus defines a foliation of $\mathscr{M}_{11}$. We also see by definition that F is a basic 4-form and because it is closed it must belong to the fourth basic cohomology class, i.e.,

$$F \in H^4_{B_\mathcal{F}},$$

where $\mathcal{F}$ is the M5-brane.

For the M2-brane it is instead $*H$ that is basic and closed and thus define the 3-dimensional foliation of the membrane.

3.2 Distributions on riemannian manifolds

We will now proceed to see what structure distributions can have after we have added a metric but first let us introduce notations regarding mappings with the metric tensor.

Definition 3.21
When considering the metric, g, and its inverse as isomorphic mappings from the tangent space into the cotangent space and vice versa, $g : T\mathscr{M} \to T^*\mathscr{M}$. We will use the standard musical notation, i.e.,

$${}^\flat X := g(X) \in T^*\mathscr{M}, \quad X \in T\mathscr{M},$$

$${}^\sharp \varphi := g^{-1}(\varphi) \in T\mathscr{M}, \quad X \in T^*\mathscr{M}.$$

We will also need the metric splitting of a two-tensor, so let us introduce notation for this.

Definition 3.22
We will define the metric trace, the anti-symmetrizer and the symmetrizer on (2,0) tensors by

$$\begin{aligned} \mathrm{tr}T &:= T(E_a, {}^\sharp E^a) \\ \wedge T(X,Y) &:= \frac{1}{2}(T(X,Y) - T(Y,X)) \\ \odot T(X,Y) &:= \frac{1}{2}(T(X,Y) + T(Y,X)) \end{aligned}$$

When structuring distributions we need the Levi–Civita connection, *i.e.*, the unique metric and torsionfree connection. So let us start by defining it, perhaps in an unfamiliar way.

Definition 3.23
Let $\mathscr{M}$ be a riemannian or pseudo-riemannian manifold with non-degenerate metric, g, then the Levi–Civita connection is the unique torsionfree connection defined by its action on a 1-form

$$\nabla\varphi(X,Y) := \frac{1}{2}(d\varphi(X,Y) + \mathscr{L}_{\sharp\varphi} g(X,Y))$$

The more familiar coordinate expression can easily be recovered by taking the coordinate vectors for X, Y and the coordinate differential for φ.

We are now ready to define the deformation tensor related to every distribution on a manifold with metric.

Definition 3.24
Let $\mathcal{D}$ be a k-distribution with projection $\mathcal{P}$ on a riemannian manifold $\mathscr{M}$ with non-degenerate metric g. Let ∇ be the Levi–Civita connection with respect to this metric and let $\mathcal{P}' := 1 - \mathcal{P}$ be the coprojection of $\mathcal{D}$. Now define the following tensors with characteristics

$$\begin{aligned} H, L, K : &\quad \Lambda^1_{\mathcal{D}} \times \Lambda^1_{\mathcal{D}} \longmapsto \Lambda^1_{\mathcal{D}'} \\ \kappa : &\quad \Lambda^1_{\mathcal{D}'} \longmapsto \mathbb{R} \end{aligned}$$

and

(i)	$H(X,Y) := \mathcal{P}'\nabla_{\mathcal{P}X}\mathcal{P}Y$	**deformation tensor**,
(ii)	$L := \wedge H$	**twisting tensor**,
(iii)	$K := \odot H$	**extrinsic curvature tensor**,
(iv)	${}^\sharp\kappa := \mathrm{tr}H$	**mean curvature tensor**,
(v)	$W := K - \frac{1}{k}{}^\sharp\kappa g$	**conformal curvature tensor**.

This gives us the decomposition of the deformation tensor in its anti-symmetric, symmetric-traceless and trace parts accordingly,

$$H = L + W + \frac{1}{k}{}^\sharp\kappa g.$$

These are the definitions of the fundamental tensors of a distribution, and we would like to make some comments about them. We see that the deformation tensor can be split into the usual anti-symmetric, symmetric-traceless and trace parts. So what does these parts say? If we just see them as generators of matrix algebras we know that these splitting refers to one tensor, the twisting tensor, that generates rotations, one tensor that generates deformations but leaves the volume constant, the conformal curvature tensor, and one tensor that scales the volume, the mean curvature tensor. In the case of distributions this is the same although we now talk about how the distribution changes while going in normal directions. From some important relations that we will see this becomes evident. But first we need a relation to prove them.

Lemma 3.25
Let $X, Y, Z \in \Lambda^1$ be vector fields on $\mathscr{M}$ with metric g. Let ∇_X be the Levi–Civita connection on $(\mathscr{M}, g)$ then

$$(\mathscr{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X).$$

proof: By direct calculation

$$\begin{aligned}(\mathscr{L}_X g)(Y, Z) =& X[g(Y, Z)] - g([X, Y], Z) - g(Y, [X, Z]) = \\ =& X[g(Y, Z)] - g(\nabla_X Y - \nabla_Y X, Z) - g(Y, \nabla_X Z - \nabla_Z X) = \\ & \{g(\nabla_X Y, Z) = X[g(Y, Z)] - g(Y, \nabla_X Z)\} \\ =& g(\nabla_Y X, Z) + g(Y, \nabla_Z X).\end{aligned}$$

■

So we get the important relations.

Proposition 3.26
Let $\mathcal{D}$ be a distribution on a manifold $\underline{\mathscr{M}}$ with metric $\underline{g}$, let further $g(X, Y) = \underline{g}(\mathcal{P}X, \mathcal{P}Y)$ be the induced metric on the distribution, then the symmetric part of the deformation tensor can be written like

$$K(X, Y)(\varphi) = -\frac{1}{2}\mathscr{L}_{\sharp\varphi'} g(X, Y), \text{ or } \quad {}^\flat K(X, Y, Z) = -\frac{1}{2}\mathscr{L}_{Z'} g(X, Y),$$

where the prime denotes projection along the normal directions by $\mathcal{P}'$. The relation for the anti-symmetric part on the other hand is

$$L(X, Y) = \frac{1}{2}\mathcal{P}'[\mathcal{P}X, \mathcal{P}Y]$$

proof: By direct calculation,

$$\begin{aligned}-\frac{1}{2}\mathscr{L}_{\sharp\varphi'} g(X, Y) =& -\frac{1}{2}(\underline{g}(\nabla_{\mathcal{P}X}\mathcal{P}'^\sharp\varphi, \mathcal{P}Y) + \underline{g}(\mathcal{P}X, \nabla_{\mathcal{P}Y}\mathcal{P}'^\sharp\varphi)) = \\ =& \frac{1}{2}(\underline{g}(\mathcal{P}'^\sharp\varphi, \mathcal{P}'\nabla_{\mathcal{P}X}\mathcal{P}Y) + \underline{g}(\mathcal{P}'\nabla_{\mathcal{P}Y}\mathcal{P}X, \mathcal{P}'^\sharp\varphi)) = \\ =& \frac{1}{2}(H(X, Y)(\varphi) + H(X, Y)(\varphi)) \\ =& K(X, Y)\end{aligned}$$

and

$$
\begin{aligned}
L(X,Y) =& \frac{1}{2}(H(X,Y) - H(Y,X)) = \\
=& \frac{1}{2}(\mathcal{P}'\nabla_{\mathcal{P}X}\mathcal{P}Y - \mathcal{P}'\nabla_{\mathcal{P}Y}\mathcal{P}X) = \\
=& \frac{1}{2}\mathcal{P}'[\mathcal{P}X, \mathcal{P}Y].
\end{aligned}
$$

■

Now it is evident that the twisting tensor which can be regarded as rotations of the distributions while going in the normal directions in fact measures how far the distribution is from being integrable. So we get a natural proposition from this.

Proposition 3.27
Let $\mathcal{D}$ be a distribution on a manifold, then $\mathcal{D}$ defines a foliation if and only if $\mathcal{D}$ is integrable, which on a riemannian manifold is equivalent to the vanishing of the tensor L above.

proof: $L = 0 \Rightarrow \mathcal{P}'[\mathcal{P}X, \mathcal{P}Y] = 0 \Rightarrow [X,Y] \in \Lambda^1_{\mathcal{D}}, \forall X, Y \in \Lambda^1_{\mathcal{D}} \Rightarrow \mathcal{D}$ *integrable*. Now 3.5 completes the proof. ■

For the case of the extrinsic curvature we see from 3.26 that it indeed measures the change of the induced metric on the distribution while going in normal directions. If we look at conformal transformations we can see that the conformal curvature tensor does not see volume changes.

Proposition 3.28
Let $\underline{\mathscr{M}}$ be a riemannian manifold with metric $\underline{g}$, let I be an almost product structure on $\underline{\mathscr{M}}$ which split the metric in $\underline{g} = g + g'$ and let $\lambda = e^{2\phi}$ be a conformal transformation on g, i.e., ${}^c\underline{g} = \lambda\underline{g}$ then the symmetric parts of the deformation tensor will transform like

$$
\begin{aligned}
{}^cK(\varphi) &= K(\varphi) + \lambda^{-1}\sharp\varphi'[\lambda]g = K(\varphi) + 2\sharp\varphi[\phi]g \\
{}^c\kappa(X) &= \kappa(X) + k\lambda^{-1}X'[\lambda] = \kappa(X) + 2kX'[\phi] \\
{}^cW &= W \\
{}^cL &= L
\end{aligned}
$$

proof: By direct calculation. ■

If we put all this together we see that we indeed have 8 fundamental classes of distributions on a riemannian manifold.

Definition 3.29
Let $\mathcal{D}$ be a distribution on a riemannian manifold $\mathscr{M}$ we have the following 8

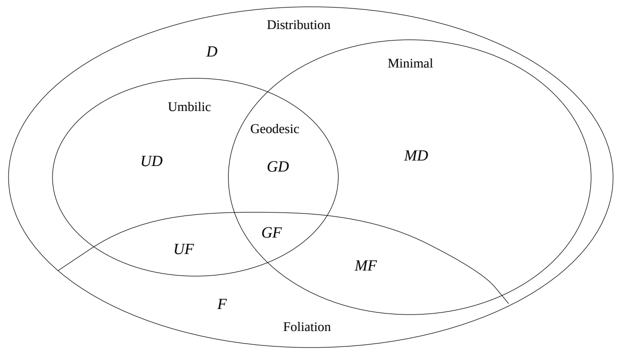

Figure 2: Overview of the different classes of a distribution

different classes

Name	$L = 0$	$W = 0$	$\kappa = 0$	*Notation*
Distribution				D
Minimal Distribution			x	MD
Umbilic Distribution		x		UD
Geodesic Distribution		x	x	GD
Foliation	x			F
Minimal Foliation	x		x	MF
Umbilic Foliation	x	x		UF
Geodesic Foliation	x	x	x	GF

4 Foliations defined by (1,1) tensors

We will in this section show how foliations can be described by certain types of endomorphisms on the tangent bundle. To start with we will therefore review the concepts of endomorphisms on the tangent bundle. In here we will see that there appears a fundamental tensor known as the Nijenhuis tensor which could be seen as the curvature of the endomorphism. We will derive this tensor in a different way from the ordinary one. This way of looking at the Nijenhuis tensor will put it on an equal basis to that of curvatures from connections on fiber bundles.

4.1 Endomorphisms on the tangent bundle

In this subsection we will look at endomorphisms, which basically are maps from the tangent bundle into itself. These maps can be described by (1,1) tensors and can equivalently be regarded as maps from the cotangent bundle to itself. We will in this section depend a lot from the results in section 2, *i.e.*, we will need the concepts of generalized Lie derivation and we will need the Nijenhuis-Föhlicher bracket which plays a central part in the study of the Nijenhuis tensor. But let us now first define the basic structure of endomorphisms.

Definition 4.1
Let $\mathscr{M}$ be a manifold and $T\mathscr{M}$ its tangent bundle, then an endomorphism I is a map

$$I : T\mathscr{M} \longmapsto T\mathscr{M}$$

i.e., it is a (1,1) tensor acting on the tangent bundle. Let $X \in \Lambda^1$ be vector field on $\mathscr{M}$ then we denote the action of I on X by

$$I : X \longmapsto I(X) = IX$$

which is nothing but the exterior product mapping defined in 2.6 i.e.,

$$\varepsilon_I X = IX$$

and can be extended, to an arbitrary p-vector, with characteristics

$$\varepsilon_I : \Lambda \longmapsto \Lambda, \quad \Lambda^p \longmapsto \Lambda^p$$

and action for $X_i \in \Lambda^1$

$$\varepsilon_I(X_1 \wedge \ldots \wedge X_p) = \sum_i X_1 \wedge \ldots \wedge IX_i \wedge \ldots \wedge X_p$$

We start by noticing that $\varepsilon_I \in \mathrm{Der}\Lambda$ is a derivation on the set of multivectors on $\mathscr{M}$. We also know that it maps vectors to vectors, which immediately lead us to the thought of associating a new bracket to this endomorphism by just taking the commutator of the respective maps.

Definition 4.2
The bracket associated with an endomorphism I is called I-bracket and denoted by $[\cdot,\cdot]_I$. It has the characteristics of a normal bracket i.e.,

$$[\,\cdot\,,\,\cdot\,]_I : \Lambda^1 \times \Lambda^1 \longmapsto \Lambda^1,$$

if $X, Y \in \Lambda^1$ are two vector fields it is defined by

$$[X,Y]_I := [IX,Y] + [X,IY] - I[X,Y]$$

and is thus manifestly antisymmetric. We also see from 2.24 that

$$[X,Y]_I \equiv \mathscr{L}_I(X \wedge Y)$$

We see that if I is the identity map the I-bracket reduces to the usual bracket. Because of this we will denote

$$\partial_I := \mathscr{L}_I = [\partial, \varepsilon_I]$$

in the case when I is an (1,1) tensor and we see that $\partial_1 = \partial$. Now ∂_I can act on a q-vector of arbitrary degree, in which the characteristics of the map looks like

$$\partial_I : \Lambda \longmapsto \Lambda, \quad \Lambda^q \longmapsto \Lambda^{q-1}.$$

This new bracket has indeed the properties of a usual vector bracket, *i.e.*, it is a anti-symmetric, non-tensorial map taking two vectors into one. The non-tensoriality looks like

$$[X, fY]_I - f[X,Y]_I = IX[f]Y,$$

and is thus depending on the directional derivative of f along IX instead of X as in the ordinary bracket. Now the original vector bracket is a Lie bracket, *i.e.*, it fulfills the Jacobi identity. One question that immediately arises is if the I-bracket is a Lie bracket. Generically the answer to this question is no. There are however cases when indeed this I-bracket is a Lie bracket, so we need a measure which tells when this is the case, and this measure will be the Nijenhuis tensor.

Definition 4.3
Let I be an endomorphism on $\mathscr{M}$, then define the **Nijenhuis tensor** *as the failure of the I-bracket to be a Lie bracket, i.e., let $X, Y, Z \in \Lambda^1$ be vector fields, then the Nijenhuis tensor is a map with characteristics*

$$\check{N}_I(X,Y,Z) : \Lambda^1 \times \Lambda^1 \times \Lambda^1 \longmapsto \Lambda^1,$$

so it is a (3,1) tensor and it measures the failure of the I-bracket in fulfilling the Jacobi identities. It is defined by

$$\check{N}_I(X,Y,Z) := [[X,Y]_I, Z]_I + [[Y,Z]_I, X]_I + [[Z,X]_I, Y]_I$$

The Nijenhuis tensor can through the equality

$$\check{N}_I(X,Y,Z) \equiv \partial_I \circ \partial_I (X \wedge Y \wedge Z)$$

be seen as measuring the failure of ∂_I to be a boundary operator.

Remark 4.4
As ∂ is not a well defined boundary operator, ∂_I will of course not be well defined either, not even when it is closed. The reason for denoting the Nijenhuis tensor with a check is that it turns up in a more natural way as a (2,1) tensor why we reserve the notation N_I to this case.

It can easily be proved that the Nijenhuis tensor defined above indeed is a tensor, *i.e.*, multilinear. We saw that the Nijenhuis tensor measured in what amount the I-bracket failed in fulfilling the Jacobi identities. This is the same as to say that the Nijenhuis tensor measures to what extent the I-bracket fails to be a Lie bracket. The conclusion is that the Nijenhuis tensor measures in what extent the endomorphism I fails to be a Lie algebra homomorphism on the infinite-dimensional Lie algebra of vector fields on $\mathscr{M}$. This conclusion will be more transparent when we introduce the other type of Nijenhuis tensor originating from the treatment of endomorphisms on the cotangent bundle.

Definition 4.5
Let $\mathscr{M}$ be a manifold and T^*M its cotangent space, let I be an endomorphism of the tangent bundle, then I^t is the natural extension characterized by

$$I^t : T^*\mathscr{M} \longmapsto T^*\mathscr{M}.$$

Let $\omega \in \Omega^1$, then the action of I^t on ω looks like

$$I^t : \omega \longmapsto I^t(\omega) = \omega I,$$

i.e., I acts as a right mapping on ω. It is nothing but the interior product of I on a 1-form and it generalizes as the dual map of the exterior product. So for $\omega \in \Omega^p$ we get

$$\begin{aligned} i_I\omega(X_1, \ldots, X_p) := \omega(\varepsilon_I(X_1 \wedge, \ldots, \wedge X_p)) = \\ \sum_i \omega(X_1, \ldots, IX_i, \ldots, X_p), \end{aligned}$$

and we find that i_I is an algebraic derivation of degree 0 with characteristics

$$i_I : \Omega \longmapsto \Omega, \quad \Omega^p \longmapsto \Omega^p$$

We do equally know in this case that $i_I \in \mathrm{Der}\Omega$ is a derivation on the cotangent bundle and it will therefore be natural to introduce the commutator of i_I and the exterior derivative in an analogous way as we introduced the I-bracket.

Definition 4.6
Let I be an endomorphism on a manifold $\mathscr{M}$ and define the associated exterior derivative, denoted by d_I, with characteristics

$$d_I : \Omega \longmapsto \Omega, \quad \Omega^p \longmapsto \Omega^{p+1}$$

by the commutator

$$d_I := [i_I, d]$$

which now is the dual map to ∂_I. Let $\omega \in \Omega^p$ be a p-form, then from 2.29 we see

$$\begin{aligned} d_I\omega(X_1, \ldots, X_p) &= \sum_i (-1)^{i+1} \mathscr{L}_{IX_i}\omega(X_1, \overset{i}{\ldots\vee\ldots}, X_p) - \\ &\quad -\omega(\partial_I(X_1 \wedge \ldots \wedge X_p)) \end{aligned}$$

or in I-bracket notation

$$\begin{aligned} d_I\omega(X_1, \ldots, X_{p+1}) &= \sum_i (-1)^{i+1} IX_i[\omega(X_1, \overset{i}{\ldots\vee\ldots}, X_{p+1})] - \\ &\quad - \sum_{i<j} (-1)^{i+j+1} \omega([X_i, X_j]_I, X_1, \overset{i}{\ldots\vee\ldots} \overset{j}{\ldots\vee\ldots}, X_{p+1}) \end{aligned}$$

Now we would like to ask whether this new operator, with the same mapping characteristics as the exterior derivative, is a coboundary operator or not, *i.e.*, whether it is nilpotent or not. So in analogy to the treatment of the I-bracket we introduce a new type of Nijenhuis tensor which measures to what extent the associated exterior derivative d_I fails in being nilpotent.

Definition 4.7
Let I be an endomorphism on $\mathscr{M}$ and define the **Nijenhuis tensor** *as the measure of how much d_I fails to be a coboundary operator. The Nijenhuis tensor is thus a (2,1) tensor. Let $X, Y \in \Lambda^1$ be vector fields on $\mathscr{M}$, then the characteristics of the Nijenhuis tensor are*

$$N_I(X,Y) : \Lambda^1 \times \Lambda^1 \longmapsto \Lambda^1$$

and we define it through the quadratic action of d_I on functions $f \in C^\infty(\mathscr{M})$,

$$< -N_I(X,Y), df >:= d_I d_I f(X,Y).$$

As we see the Nijenhuis tensor measures the failure in closure of the operator d_I and can thus be considered as a form of torsion. Alternatively, as the below equivalent definition shows, it measures the curvature of the endomorphism, i.e.,

$$N_I[X,Y] := I([X,Y]_I) - [I(X), I(Y)],$$

so the Nijenhuis tensor can be seen as measuring how far this endomorphism is from being a Lie algebra homomorphism of the infinite-dimensional Lie algebra of vector fields on $\mathscr{M}$.

proof: The proof follows from definition 4.2 and 2.31. ■

Remark 4.8
Notice that the expression for the Nijenhuis tensor in definition 4.7 differs by a sign from the original definition. This definition turns out to be more natural in two different aspects. First of all we find that it looks similar to the curvature of algebraic gauge theory and further we see that if ∇ is a connection with torsion on $\mathscr{M}$, then $\nabla \wedge \nabla f(X,Y) =< -T(X,Y), df >$. We will show later that the Nijenhuis tensor can in fact be viewed as a kind of torsion, which makes the new sign natural.

When we write the Nijenhuis tensor on the above form the connection to algebraic gauge theory is clear. In algebraic gauge theory we have a principal bundle $0 \to A \to E \to B \to 0$ and a connection $\rho : B \to E$ with curvature

$$F(X,Y) := \rho([X,Y]|_B) - [\rho(X), \rho(Y)]|_E.$$

The curvature therefore measures to what extent ρ fails to be a Lie algebra homomorphism. The conclusion is that the Nijenhuis tensor describes curvatures in principal bundles. We will look at this more thoroughly later, but first we will examine some basic relations involving the Nijenhuis tensor that will be needed in the sequel. We start with a small lemma.

Lemma 4.9
Let I, I_1, I_2 be endomorphisms on $\mathscr{M}$ and let $X, Y \in \Lambda^1$ be two vector fields, then

$$\begin{aligned} (i) \qquad & \mathscr{L}_X I(Y) = [X, IY] - I[X,Y] \\ (ii) \qquad & [I_1, I_2](X,Y) = [I_1X, I_2Y] + [I_2X, I_1Y] - I_1[X,Y]_{I_2} - I_2[X,Y]_{I_1} \end{aligned}$$

proof: (i) by direct calculation $\mathscr{L}_X I(Y) = \mathscr{L}_X(IY) - I(\mathscr{L}_X Y) = [X, IY] - I[X, Y]$ and
(ii) directly from definition 2.31. ∎

Proposition 4.10
Let $I,\ I_1,\ I_2$ be endomorphisms on $\mathscr{M}$ and let $X, Y \in \Lambda^1$ be two vector fields, then we have the following relations involving the Nijenhuis tensor.

$$
\begin{aligned}
&(i) && N_I(X,Y) = (I\mathscr{L}_X I - \mathscr{L}_{IX} I)(Y) \\
&(ii) && N_I = -\frac{1}{2}[I, I] \\
&(iii) && N_{\lambda I} = \lambda^2 N_I \\
&(iv) && N_{I_1+I_2} = N_{I_1} + N_{I_2} - [I_1, I_2]
\end{aligned}
$$

proof: (i) follows from the first part of lemma 4.9 while (ii), (iii) and (iv) is a direct consequence of the properties of the bracket in lemma 4.9. ∎

We will also list some properties involving the identity endomorphism which as expected turns out to be trivial.

Proposition 4.11
Let I be a endomorphism on $\mathscr{M}$ and 1 the identity operator (endomorphism), then we have the following relations involving the Nijenhuis tensor

$$
\begin{aligned}
&(i) && [1, I] = 0 \\
&(ii) && N_{1+I} = N_I
\end{aligned}
$$

proof: Trivial. ∎

Now we have defined two types of Nijenhuis tensors, one as the natural one occurring on the space of p-vectors and the other appearing on the space of differential forms. Of course there will be no surprise to us that these two types of tensors in fact are related. This relation will be seen in following proposition.

Proposition 4.12
Let $X,\ Y,\ Z \in \Lambda^1$ be vector fields on $\mathscr{M}$ and let $\tilde{N}_I$ be the Nijenhuis tensor defined in 4.3 and N_I be the one defined in 4.7. These are then related as

$$\tilde{N}_I(X,Y,Z) = \sum_{cycl} N_I([X,Y],Z) + [N_I(X,Y),Z]$$

proof:

By definition 2.31 and proposition 2.24 we get

$$\begin{aligned}\tilde{N}_I(X,Y,Z) &= \partial_I\partial_I(X\wedge Y\wedge Z)\\ &= \frac{1}{2}[\check{\mathscr{L}}_I,\check{\mathscr{L}}_I](X\wedge Y\wedge Z)\\ &= -\check{\mathscr{L}}_{\frac{1}{2}[I,I]}(X\wedge Y\wedge Z)\\ &= \check{\mathscr{L}}_{N_I}(X\wedge Y\wedge Z)\\ &= \sum_{cycl} N_I([X,Y],Z)+[N_I(X,Y),Z]\end{aligned}$$

■

We also have the complete relation when d_I acts on an arbitrary differential form, which follows.

Proposition 4.13
Let I be an endomorphism on $\mathscr{M}$ and d_I be its associated coboundary operator, let ∂_I be the formal boundary operator associated with the endomorphism, then we have

$$\begin{aligned}d_I d_I\omega(X_1,\dots,X_{p+2}) &= \sum_{i<j}(-1)^{i+j}\mathscr{L}_{N_I(X_i,X_j)}\omega(X_1,\overset{i}{\checkmark}\dots\overset{j}{\checkmark}\dots,X_{p+2})+\\ &\quad \omega(\partial_I\partial_I(X_1\wedge\ldots\wedge X_{p+2}))\end{aligned}$$

proof: By 2.31 and 2.29, using $d_I d_I=\frac{1}{2}[\hat{\mathscr{L}}_I,\hat{\mathscr{L}}_I]=-\hat{\mathscr{L}}_{N_I}$. ■

4.1.1 Manifolds with metric

If we add to the manifold the structure of a non-degenerate metric, we are able to introduce a Levi–Civita connection and we can in a similar fashion as above introduce a new bracket structure, the Jordan bracket.

Definition 4.14
Let $\underline{\mathscr{M}}$ be a riemannian manifold with Levi–Civita connection $\underline{\nabla}$, then define the **Jordan bracket**, *denoted by $\{\cdot,\cdot\}$, with the following characteristics:*

$$\{\,\cdot\,,\,\cdot\,\}:\Lambda^1\times\Lambda^1\longmapsto\Lambda^1,$$

by

$$\{X,Y\}:=\underline{\nabla}_X Y+\underline{\nabla}_Y X$$

where $X,Y\in\Lambda^1$ are vector fields on $\underline{\mathscr{M}}$. Now define the Jordan bracket associated with an endomorphism I, denoted $\{\cdot,\cdot\}_I$, in an analogous fashion to $[\cdot,\cdot]_I$, by

$$\{X,Y\}_I:=\{IX,Y\}+\{X,IY\}-I\{X,Y\}.$$

We see that the Jordan bracket associated with an endomorphism is defined in a similar fashion as the I-bracket was earlier. It should also be pointed out that the Jordan bracket and the usual vector bracket of two vectors, $X, Y \in \Lambda^1$ in fact only measure the symmetric and anti-symmetric parts of the tensor $\nabla_X Y$. We can also introduce the Jordan tensor in the same way as we did with the Nijenhuis tensor.

Definition 4.15
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define a riemannian almost product structure, and let $\{\cdot,\cdot\}$ be the Jordan bracket, then define the **Jordan tensor** *associated to I, denoted M_I, with the following characteristics:*

$$M_I : \Lambda^1 \times \Lambda^1 \longmapsto \Lambda^1,$$

by

$$M_I(X, Y) := I\{X, Y\}_I - \{IX, IY\}$$

where $X, Y \in \Lambda^1$ are vector fields on $\underline{\mathscr{M}}$. The analogy to the Nijenhuis tensor is obvious .

So we see that the Jordan tensor measures the failure of the Jordan bracket to commute with the endomorphism, I. We also get similar relations for the Jordan tensor as for the Nijenhuis tensor earlier.

Proposition 4.16
Let $I,\ I_1,\ I_2$ be endomorphisms on $\underline{\mathscr{M}}$ and let $X, Y \in \Lambda^1$ be two vector fields. Introduce the operator $\mathscr{T}_X := \underline{\nabla}_X - \mathscr{L}_X$, then we have the following relations involving the Jordan tensor.

$$\begin{aligned} (i) \qquad & M_I(X, Y) = (I\mathscr{T}_X I - \mathscr{T}_{IX} I)(Y) \\ (ii) \qquad & M_I = -\frac{1}{2}\{I, I\} \\ (iii) \qquad & M_{\lambda I} = \lambda^2 M_I \\ (iv) \qquad & M_{I_1+I_2} = M_{I_1} + M_{I_2} - \{I_1, I_2\} \end{aligned}$$

In short, the Nijenhuis tensor measures the non-commutativity between an endomorphism I and the antisymmetric part of the Levi–Civita connection, while the Jordan tensor measures the non-commutativity between an endomorphism I and the symmetric part of the Levi–Civita connection. As the anti-symmetric part of the Levi–Civita connection is nothing but the usual vector bracket (or the exterior derivative if seen as acting on forms), we note that the Nijenhuis tensor is independent of the metric and thus definable even without a metric present. This has been commented on earlier. If we now have a metric, the two structures can be combined naturally to form the deformation tensor associated with an endomorphism.

Definition 4.17
Let I be an endomorphism on a manifold $\mathscr{M}$ with non-degenerate metric, g. Let ∇ be the Levi–Civita connection on $\mathscr{M}$ and define the **deformation tensor**

associated with the endomorphism I, denoted H_I, with the following characteristics:

$$H_I : \Lambda^1 \times \Lambda^1 \longmapsto \Lambda^1.$$

H_I is defined by the expression

$$H_I(X,Y) := (I\nabla_X I - \nabla_{IX} I)(Y),$$

where $X, Y \in \Lambda^1$ are two vector fields on $\mathscr{M}$. We immediate note the equivalent definition

$$H_I(X,Y) := N_I(X,Y) + M_I(X,Y).$$

We will later see that in the case where the endomorphism I is a riemannian almost product structure the deformation tensor will be analogous to that in the earlier section.

4.2 Foliations from endomorphisms

The preceding sections give us the opportunity to formulate the concepts of distributions and foliations in the framework of a special type of endomorphism on the tangent bundle. We will see that the type of endomorphism will be very similar to that of an almost complex structure. But to start with we will change our notation a bit in order to get a more compact language when considering distributions on a manifold.

Notation 4.18
We will denote the objects on our space with an underline, i.e.,

$\underline{\mathscr{M}}$	*Manifold*
$T\underline{\mathscr{M}}$	*Tangent bundle of $\underline{\mathscr{M}}$*
$T^*\underline{\mathscr{M}}$	*Cotangent bundle of $\underline{\mathscr{M}}$*
$\underline{g}$	*Metric on $\underline{\mathscr{M}}$*
$\underline{d}$	*Exterior derivative*
$\underline{X}$	*Vector field on $\underline{\mathscr{M}}$*

to list the primarily used objects. We will use this underlining principle for all objects on $\underline{\mathscr{M}}$ whenever there may be a risk of confusion.

When considering endomorphisms in the preceding subsection, where we defined the Nijenhuis tensor, we were treating endomorphisms in the most general sense and had no conditions on the endomorphism I at all. But there are of course certain types of endomorphisms that are more interesting than others. In mathematics there are four basic types which are of great importance. We will define them below.

Definition 4.19
Let I be an endomorphism tensor of type (1,1), i.e., it maps $T\underline{\mathscr{M}} \to T\underline{\mathscr{M}}$ or $T^*\underline{\mathscr{M}} \to T^*\underline{\mathscr{M}}$ then I is called

$$\begin{array}{rll} (i) & \textbf{Nilpotent}, if & I^2 = 0, \\ (ii) & \textbf{Idempotent}, if & I^2 = I, \\ (iii) & \textbf{Almost product structure}, if & I^2 = 1, \\ (iv) & \textbf{Almost complex structure}, if & I^2 = -1. \end{array}$$

Of course the concepts nilpotent and idempotent could be generalized to hold for a different power than 2, but otherwise these are the four basic types. Interesting to note is that for a nilpotent endomorphism $\ker I \subset \mathrm{Im} I$ which implies that $\mathrm{rank} I \leq [n/2]$. For an idempotent endomorphism the rank can be arbitrary. The last two types of endomorphisms which are called almost product (complex) structures are both of full rank. In this section we will see that an almost product structure will be just the kind of endomorphism that one needs in the theory of distributions and foliations. Although the study of almost product structures could take place without introducing a metric on the manifold, we will focus on the treatment of manifolds with a metric. We will only point out that, as seen in previous subsection, all structure involving only the Nijenhuis tensor exist even without metrics. But let us now introduce a metric on the manifold.

Definition 4.20
Let I be an almost product structure on a manifold $\underline{\mathscr{M}}$ with riemannian metric $\underline{g}$ and let $X,\ Y \in T\underline{\mathscr{M}}$ be vector fields. Then the triplet ($\underline{\mathscr{M}}$, $\underline{g}$, I) is called an riemannian almost product structure if

$$\underline{g}(IX, IY) = \underline{g}(X, Y)$$

or in other words, I is a automorphism of $\underline{g}$ in the sense that the following diagram commutes:

$$\begin{tikzcd} T\underline{\mathscr{M}} \arrow[r, "I"] \arrow[d, "\underline{g}"'] & T\underline{\mathscr{M}} \arrow[d, "\underline{g}"] \\ T^*\underline{\mathscr{M}} \arrow[r, "I^t"'] & T^*\underline{\mathscr{M}} \end{tikzcd}$$

i.e.,

$$I^t \circ \underline{g} \circ I = \underline{g}$$

So we see that the endomorphism I conserves the length of a vector. This immediately tells us that it must be a local $O(\underline{m})$ transformation on the tangent bundle. In the above definition we required that the metric satisfied $\underline{g}(X, Y) = \underline{g}(IX, IY)$ but of course from any riemannian metric not satisfying this we can always construct a new one just by taking $\tilde{\underline{g}}(X, Y) := \underline{g}(X, Y) + \underline{g}(IX, IY)$ which would then satisfy the above condition. Therefore, this implies no restriction on the manifold.

We will now look at the properties of a riemannian almost product structure in a little more detail to find out in what sense it defines distributions on the manifold.

Proposition 4.21
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define a riemannian almost product structure on $\underline{\mathscr{M}}$ with $\dim \underline{\mathscr{M}} = \underline{m}$, then

(i). I is a local $O(\underline{m})$ matrix with $\frac{1}{2}\underline{m}(\underline{m}-1)$ independent components.

(ii). All eigenvalues are ± 1.

(iii). $\mathrm{tr} I = 2k - \underline{m}$*, where k is the number of positive eigenvalues.*

(iv). There is a preferred base called the oriented base in which I is diagonal and ordered, i.e., it takes the form

$$I = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$$

proof: (i) Let $\{E_{\underline{a}}\}$ be an orthonormal frame, then $\eta_{\underline{ab}} = \underline{g}(E_{\underline{a}}, E_{\underline{b}}) = \underline{g}(IE_{\underline{a}}, IE_{\underline{b}}) = I_{\underline{a}}{}^{\underline{c}} I_{\underline{b}}{}^{\underline{d}} \eta_{\underline{cd}} \Rightarrow I \in O(\underline{m})$. (ii) $1 = I^2 = PDP^{-1}PDP^{-1} \Rightarrow D^2 = 1 \Rightarrow$ all eigenvalues ± 1. (iii) $\mathrm{tr} I = k + (-1)(\underline{m} - k) = 2k - \underline{m}$. (iv) Let $\{E_{\bar{a}} = (E_a, E_{a'})\}$ be a oriented and ordered base, then $I = E^a E_a - E^{a'} E_{a'}$. ■

These properties of an almost product structure tells us that if we express our vectors in terms of the eigenvectors of I (preferably in the oriented base) I acts as reflecting the vectors in the hyperplane spanned by the eigenvectors with positive eigenvalue 1. Now the set of vectors lying in this hyperplane will be invariant under I while those lying in the normal hyperplane will change to the opposite direction under I. One can say that I breaks the structure group $O(\underline{m})$ of $T\underline{\mathscr{M}}$ down to $O(k) \times O(k')$ where $k' := \underline{m} - k$. In that sense the set of almost product structures with k positive eigenvalues is parameterized by the grassmannian,

$$I \in Gr(k, \underline{m}) = \frac{O(\underline{m})}{O(k) \times O(k')} \tag{1}$$

The grassmannian has $kk' = k(\underline{m} - k)$ independent components and parameterizes the space of k-planes in $\mathbb{R}^{\underline{m}}$. We can now let an almost product structure define two complementary distributions for us by taking these complementary hyperplanes spanned by the eigenfunctions with positive eigenvalues and by the eigenfunctions with negative eigenvalues respectively.

Definition 4.22
Let I be an almost product structure on $\underline{\mathscr{M}}$, then I defines two natural distributions of $T\underline{\mathscr{M}}$, denoted $\mathcal{D}$ and $\mathcal{D}'$ respectively, in the following way. Let

$$\begin{aligned} \mathcal{D}_x &:= \{X \in T_x\underline{\mathscr{M}} : IX = X\}, \\ \mathcal{D}'_x &:= \{X \in T_x\underline{\mathscr{M}} : IX = -X\}, \end{aligned}$$

then

$$\mathcal{D} := \bigcup_{x \in \underline{\mathscr{M}}} \mathcal{D}_x, \quad \mathcal{D}' := \bigcup_{x \in \underline{\mathscr{M}}} \mathcal{D}'_x.$$

Again it should be noted that the distributions defined above are independent of the existence of a metric on $\underline{\mathscr{M}}$. The main difference is that in the case where we have a metric the structure group of $T\underline{\mathscr{M}}$ breaks down from $GL(\underline{m})$ down to $O(\underline{m})$, and the almost product structure will thus be parameterized under the grassmannian space previously introduced. In the case where we don't have a metric the almost product structure would be parameterized under $GL(\underline{m})/(GL(k) \times GL(k'))$. As we know that the almost product structure squares to one, we can define two complementary projection operators, which of course also are endomorphisms on the tangent bundle.

Definition 4.23
From an almost product structure I on a manifold $\underline{\mathscr{M}}$ we can define two projection operators through

$$\begin{aligned} \mathcal{P} &:= \frac{1}{2}(1 + I) \\ \mathcal{P}' &:= \frac{1}{2}(1 - I). \end{aligned}$$

These will be mappings in the sense $\mathcal{P} : T\underline{\mathscr{M}} \to \mathcal{D}$ and $\mathcal{P}' : T\underline{\mathscr{M}} \to \mathcal{D}'$ respectively.

We see that we can regard the distributions $\mathcal{D}, \mathcal{D}'$ as subbundles of the tangent bundle. In this sense the projection operators take an element in $T\underline{\mathscr{M}}$ down to an element in $\mathcal{D}$ and $\mathcal{D}'$ respectively. The map is by definition surjective and if we require that the almost product structure is C^k, or even C^∞, the map will be a surjective submersion. We can also introduce canonical inclusions with respect to these submersions.

Definition 4.24
We can define the canonical inclusions $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}}'$ of $\mathcal{F}$ and $\mathcal{F}'$ in $T\underline{\mathscr{M}}$ by the commutativity of the following diagram:

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\tilde{\mathcal{P}}"] \arrow[d, "Id"'] & T\underline{\mathscr{M}} \arrow[dl, "\mathcal{P}"] \arrow[dr, "\mathcal{P}'"'] & \mathcal{D}' \arrow[l, "\tilde{\mathcal{P}}'"'] \arrow[d, "Id"] \\ \mathcal{D} & & \mathcal{D}' \end{tikzcd}$$

We see from the definition that these inclusions are equivalently defined by $\mathcal{P}\tilde{\mathcal{P}} = 1_{\mathcal{D}}, \mathcal{P}'\tilde{\mathcal{P}}' = 1_{\mathcal{D}'}$. These projection operators split the tangent bundle into two complementary parts.

Proposition 4.25
Let $\underline{\mathscr{M}}$ be a manifold, $T\underline{\mathscr{M}}$ its tangent bundle, $\mathcal{F} \subset T\underline{\mathscr{M}}$ and $\mathcal{F}' \subset T\underline{\mathscr{M}}$ two subbundles with projectors $\mathcal{P}$ and $\mathcal{P}'$ respectively, then the following statements are equivalent:

(i). $\mathcal{D} \cap \mathcal{D}' = 0, \quad \mathcal{D} \cup \mathcal{D}' = T\underline{\mathscr{M}}$

(ii). The short sequence

$$0 \longrightarrow \mathcal{D} \xrightarrow{\tilde{\mathcal{P}}} T\underline{\mathscr{M}} \xrightarrow{\mathcal{P}'} \mathcal{D}' \longrightarrow 0$$

is exact.

proof: Trivial by the exactness of the sequence $\mathcal{F}' = T\underline{\mathscr{M}}/\mathcal{F}$. ■

What this tells us is in fact that the almost product structure I in form of its associated projection operators splits the tangent bundle into

$$T\underline{\mathscr{M}} = \mathcal{D} \oplus \mathcal{D}'. \tag{2}$$

Later we will show that in the case of a principal bundle, one of the projection operators of I will in fact be the connection of the principal bundle, and the Nijenhuis tensor, N_I, will measure the curvature of this connection. An interesting point regarding these inclusion maps is that if we restrict the projection operators to a submanifold $\mathscr{M}$ of $\underline{\mathscr{M}}$ in such way that $T\mathscr{M}$ is spanned by the eigenvectors of $\mathcal{P}$, and that these furthermore are integrable, then a map $f : \mathscr{M} \mapsto \underline{\mathscr{M}}$ is in fact an embedding and $\tilde{\mathcal{P}}$ the associated embedding matrix. We will see this more transparently later, but first we will come back to the Nijenhuis tensor and investigate how its structure is affected by imposing the condition of an almost product structure to the endomorphism I.

Lemma 4.26
Let I be an almost product structure on a manifold $\underline{\mathscr{M}}$ and let its associated projection operators be $\mathcal{P} := \frac{1}{2}(1 + I)$, $\mathcal{P}' := \frac{1}{2}(1 - I)$, then

$$\begin{aligned}
&(i) & N_{\mathcal{P}} =& N_{\mathcal{P}'} \\
&(ii) & N_I =& 4N_{\mathcal{P}} \\
&(iii) & \frac{1}{2}[\mathcal{P}, \mathcal{P}'] =& N_{\mathcal{P}} \\
&(iv) & N_{\mathcal{P}}(X, Y) =& -\mathcal{P}'[\mathcal{P}X, \mathcal{P}Y] - \mathcal{P}[\mathcal{P}'X, \mathcal{P}'Y]
\end{aligned}$$

proof: By direct calculation
(i) $N_{\mathcal{P}} = \frac{1}{2}[\mathcal{P}, \mathcal{P}] = \frac{1}{2}[1 - \mathcal{P}', 1 - \mathcal{P}'] = \frac{1}{2}[\mathcal{P}', \mathcal{P}'] = N_{\mathcal{P}'}$,
(ii) $N_I = N_{\mathcal{P}-\mathcal{P}'} = N_{2\mathcal{P}-1} = 4N_{\mathcal{P}}$,

(iii) $\frac{1}{2}[\mathcal{P},\mathcal{P}'] = \frac{1}{2}[\mathcal{P},1-\mathcal{P}] = N_{\mathcal{P}}$,
(iv) -

$$\begin{aligned}
N_{\mathcal{P}}(X,Y) =&[\mathcal{P}X,\mathcal{P}Y] + \mathcal{P}^2[X,Y] - \mathcal{P}[\mathcal{P}X,Y] - \mathcal{P}[X,\mathcal{P}Y] = \\
=&[\mathcal{P}X,\mathcal{P}Y] + \mathcal{P}[X,Y] - \mathcal{P}[X,Y] + \mathcal{P}[\mathcal{P}'X,Y] - \mathcal{P}[X,\mathcal{P}Y] = \\
=&\mathcal{P}'[\mathcal{P}X,\mathcal{P}Y] + \mathcal{P}[\mathcal{P}X,\mathcal{P}Y] + \mathcal{P}[\mathcal{P}'X,\mathcal{P}Y] + \\
&+ \mathcal{P}[\mathcal{P}'X,\mathcal{P}'Y] - \mathcal{P}[\mathcal{P}X,\mathcal{P}Y] - \mathcal{P}[\mathcal{P}'X,\mathcal{P}Y] = \\
=&\mathcal{P}'[\mathcal{P}X,\mathcal{P}Y] + \mathcal{P}[\mathcal{P}'X,\mathcal{P}'Y]
\end{aligned}$$

■

From the above lemma it is clear that the Nijenhuis tensor measures to what extent the two complementary distributions, associated with an almost product structure, fail to integrable.

Proposition 4.27
Let the triplet $(\underline{\mathscr{M}},\underline{g},I)$ define an riemannian almost product structure and let L, L' be the twisting tensors of the distributions defined by I, then

$$\frac{1}{8}N_I = -L - L'.$$

proof: By 3.26 and 4.27. ■

Again it is noted that all these tensors are invariant under the metric and exist even without a metric on the manifold. We see that in case we have a foliation, proposition 3.27 tells us that at least one of the twisting tensors vanishes. This is the same as saying that $N_I(X,Y)$ is an eigenvector of I, *i.e.*, $IN_I(X,Y) = \pm N_I(X,Y)$. In the case when both associated distributions are integrable, the Nijenhuis tensor vanishes and we have two complementary foliations on the manifold. We will see later that this will lead us to the case where the exterior algebra in fact splits and becomes doubly graded under the exterior derivatives associated with complementary projections $\mathcal{P},\mathcal{P}'$ of I. But let us first see what extra structure an almost product structure will give us in the case where we indeed have a metric on the manifold. We start be noticing that to every metric $\underline{g}$ on $\underline{\mathscr{M}}$, we have two complementary metrics associated with the almost product structure.

Definition 4.28
Let $\underline{\mathscr{M}}$ be a riemannian or pseudo-riemannian manifold with metric, $\underline{g}$, I a reflective structure with $\mathcal{P}$ and $\mathcal{P}'$ the corresponding projectors, then define the two associated metrics with respect to the reflective structure by

$$g(X,Y) := \underline{g}(\mathcal{P}X,\mathcal{P}Y), \quad g'(X,Y) := \underline{g}(\mathcal{P}'X,\mathcal{P}'Y)$$

which implies that $\underline{g}$ splits into these two parts, i.e.,

$$\underline{g} = g + g'.$$

Of course we note that it is the condition $\underline{g}(IX, IY) = \underline{g}(X, Y)$ that implies that two complementary vectors are orthogonal, *i.e.*, $\underline{g}(\mathcal{P}X, \mathcal{P}'Y) = 0$. In an analogous way to which we deduced the new structure to the Nijenhuis tensor we can proceed to find out how an almost product structure reduces the Jordan tensor. We will see that due to the similar bracket structure the structure found in the Nijenhuis tensor will be similar.

Lemma 4.29
Let I be an almost product structure on a manifold $\underline{\mathscr{M}}$ and let its associated projection operators be $\mathcal{P} := \frac{1}{2}(1+I)$, $\mathcal{P}' := \frac{1}{2}(1-I)$. Let M denote the Jordan tensor, then

$$
\begin{aligned}
&(i) && M_{\mathcal{P}} = M_{\mathcal{P}'} \\
&(ii) && M_I = 4M_{\mathcal{P}} \\
&(iii) && \frac{1}{2}\{\mathcal{P}, \mathcal{P}'\} = M_{\mathcal{P}} \\
&(iv) && M_{\mathcal{P}}(X, Y) = -\mathcal{P}'\{\mathcal{P}X, \mathcal{P}Y\} - \mathcal{P}\{\mathcal{P}'X, \mathcal{P}'Y\}
\end{aligned}
$$

proof: Similar to that of 4.27. ■

As the Jordan bracket is just the symmetric part of the covariant derivative while the usual vector bracket can be regarded as the anti-symmetric part, the structure on the bracket level will of course be similar, but they do of course measure two different things. Notable is that, in contrary to the Nijenhuis tensor, the Jordan tensor makes no sense in a manifold without metric but the connection used in the Jordan bracket must be metric.

Proposition 4.30
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define a riemannian almost product structure, K, K' be the extrinsic curvature tensors of the distributions defined by I, then

$$\frac{1}{8}M_I = -K - K'.$$

proof: By lemma 4.29 and definition 3.24. ■

So, put together, we see that all the structure of two complementary distributions can be put into this single almost product structure I. The deformation tensors are recovered by the associated Nijenhuis tensor and Jordan tensor, of which the Nijenhuis tensor contains the integrability conditions while the Jordan tensor contains the extrinsic curvature parts. We will use this in the classification scheme of riemannian almost product structures in next subsection. Now however we are interested in the connection $\underline{\nabla}$. First of all it is easily proven that although annihilating the metric $\underline{g}$ it does not annihilate g, g'. We would then instead like a new connection which we will denote $\underline{\tilde{\nabla}}$ and call the adapted connection that does annihilate all these metrics so it commutes with all of them.

Definition 4.31
Let $\underline{\mathscr{M}}$ be a riemannian or pseudo-riemannian manifold with non-degenerate metric $\underline{g}$ and corresponding Levi–Civita connection $\underline{\nabla}$. Let I define a foliation in the previous sense, then the following two definitions of the adapted connection are equivalent

(i). $\underline{\tilde{\nabla}}_X Y := \underline{\nabla}_X Y + A(X,Y), \quad A(X,Y) := \frac{1}{2} I \underline{\nabla}_X I(Y)$

(ii). $\underline{\tilde{\nabla}}_X Y := \mathcal{P}\underline{\nabla}_X \mathcal{P}Y + \mathcal{P}'\underline{\nabla}_X \mathcal{P}'Y$

proof: The proof is immediate. ■

We will soon prove that this connection indeed annihilates all the metrics, and it is therefore suitable for calculations on the subbundles generated by the almost product structure I. But first we will introduce yet another connection, called the Vidal connection [25], not with the property of being metric but with additional properties which will become clear later.

Definition 4.32
Let $\underline{\mathscr{M}}$ be a riemannian or pseudo-riemannian manifold with non-degenerate metric $\underline{g}$ and corresponding Levi–Civita connection $\underline{\nabla}$, let I define a foliation in the previous sense, then the Vidal connection is defined by

$$\underline{\tilde{\tilde{\nabla}}}_X Y := \underline{\tilde{\nabla}}_X Y + B(X,Y), \quad B(X,Y) := \frac{1}{4}(\underline{\nabla}_{IY} I + I\underline{\nabla}_Y I)(X).$$

Of course both the tensors A and B will only contain parts of the deformation tensor and are in fact related.

Proposition 4.33
Let B be the tensor defined in 4.32 and A the tensor defined in 4.31, then we can express the tensor B in terms of A and the almost product structure I as

$$B(X,Y) = \frac{1}{2}\left(A(Y,X) - A(IY,IX)\right).$$

proof: By direct calculation from the definitions,

$$\begin{aligned}\frac{1}{2}\left(A(Y,X) - A(IY,IX)\right) =& \frac{1}{4}\left(I\underline{\nabla}_Y I(X) - I\underline{\nabla}_{IY} I(IX)\right) = \\ =& \frac{1}{4}(I\underline{\nabla}_Y I + \underline{\nabla}_{IY} I)(X) = \\ =& B(X,Y)\end{aligned}$$

■

We will see the total structure of these two connections last in this subsection, and we will first list a number of their fundamental properties. The most important property that both these connections satisfy is that they commute with the almost product structure.

Proposition 4.34
Let $\tilde{\underline{\nabla}}$ denote the adapted connection defined in 4.31 and $\tilde{\tilde{\underline{\nabla}}}$ the Vidal connection defined in 4.32, then their principal feature is that they both commute with the almost product structure I, i.e.,

$$\tilde{\underline{\nabla}}_X I = \tilde{\tilde{\underline{\nabla}}}_X I = 0$$

proof: (i)

$$\begin{aligned}\tilde{\underline{\nabla}}_X IY =& \underline{\nabla}_X IY + \frac{1}{2}(I\underline{\nabla}_X I)(IY) = \\ =& I\underline{\nabla}_X Y + (\underline{\nabla}_X I)(Y) - \frac{1}{2}(\underline{\nabla}_X I)(Y) = \\ =& I\underline{\nabla}_X Y + \frac{1}{2}I^2(\underline{\nabla}_X I)(Y) = I\tilde{\underline{\nabla}}_X Y\end{aligned}$$

(ii)

$$\begin{aligned}\tilde{\tilde{\underline{\nabla}}}_X IY =& \tilde{\underline{\nabla}}_X IY + \frac{1}{4}(\underline{\nabla}_{I^2Y} I + I\underline{\nabla}_{IY} I)(X) = \\ =& I\tilde{\underline{\nabla}}_X Y + \frac{1}{4}(I^2\underline{\nabla}_Y I + I\underline{\nabla}_{IY} I)(X) = \\ =& I\tilde{\tilde{\underline{\nabla}}}_X Y\end{aligned}$$

■

As already mentioned, the adapted connection is metric. This is not the case for the Vidal connection, but it is nevertheless important. Its properties will be examined in the following subsection, where the basic types of riemannian almost product structures will be classified. But let us now show that the adapted connection indeed annihilates all associated metrics. To help us out we need the following lemma.

Lemma 4.35
Let A be the tensor defined in 4.31, then we have the relation

$$\underline{g}(A(X,Y),Z) + \underline{g}(Y, A(X,Z)) = 0$$

proof: $2\underline{g}(A(X,Y),Z) = \underline{g}(I\underline{\nabla}_X I(Y), Z) = \underline{g}(\underline{\nabla}_X I(Y), IZ) =$
$\underline{g}(\underline{\nabla}_X IY - I\underline{\nabla}_X Y, IZ) = -\underline{g}(Y, I\underline{\nabla}_X IZ - \underline{\nabla}_X Z) = -\underline{g}(Y, I\underline{\nabla}_X I(Z)) =$
$-2\underline{g}(Y, A(X,Z))$ ■

Now it is straightforward to prove that the adapted connection is metric.

Proposition 4.36
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ be a riemannian almost product structure on $\underline{\mathscr{M}}$ and $\tilde{\underline{\nabla}}$ the adapted connection defined in 4.31, then this connection is metric with respect to the splitting of $\underline{g}$ according to 4.28, i.e.,

$$\tilde{\underline{\nabla}} g = 0$$
$$\tilde{\underline{\nabla}} g' = 0$$

proof: We have

$$\begin{aligned}\tilde{\underline{\nabla}}_X \underline{g}(Y,Z) =& X[\underline{g}(Y,Z)] - \underline{g}(\tilde{\underline{\nabla}}_X Y, Z) - \underline{g}(Y, \tilde{\underline{\nabla}}_X Z) = \\ =& -\underline{g}(A(X,Y),Z) - \underline{g}(Y, A(X,Z)) = 0,\end{aligned}$$

and $\tilde{\underline{\nabla}}_X I = 0 \Rightarrow \tilde{\underline{\nabla}}_X \mathcal{P} = 0$ so we see

$$\begin{aligned}(\tilde{\underline{\nabla}}_X g)(Y,Z) =& X[g(Y,Z)] - g(\tilde{\underline{\nabla}}_X Y, Z) - g(Y, \tilde{\underline{\nabla}}_X Z) = \\ =& X[\underline{g}(\mathcal{P}Y, \mathcal{P}Z)] - \underline{g}(\mathcal{P}\tilde{\underline{\nabla}}_X Y, \mathcal{P}Z) - \underline{g}(\mathcal{P}Y, \mathcal{P}\tilde{\underline{\nabla}}_X Z) = \\ =& X[\underline{g}(\mathcal{P}Y, \mathcal{P}Z)] - \underline{g}(\tilde{\underline{\nabla}}_X \mathcal{P}Y, \mathcal{P}Z) - \underline{g}(\mathcal{P}Y, \tilde{\underline{\nabla}}_X \mathcal{P}Z) = \\ =& (\tilde{\underline{\nabla}}_X \underline{g})(\mathcal{P}Y, \mathcal{P}Z) = 0\end{aligned}$$

■

We can now see in a more transparent way how the different parts of these connections look. It becomes most conceptually clear if we use the oriented base.

Proposition 4.37
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define a riemannian almost product structure, let $\underline{\omega}$, $\tilde{\underline{\omega}}$ and $\tilde{\tilde{\underline{\omega}}}$ denote the connection one-forms of the Levi–Civita connection, the adapted connection and the Vidal connection respectively. Let furthermore H, H' denote the deformation tensors with respect to I and C, C' be coefficients of anholonomy, then

$$\begin{aligned}\underline{\omega} &= \left[\begin{pmatrix} \omega & H \\ -H^t & \Omega \end{pmatrix}, \begin{pmatrix} \Omega' & H' \\ -H'^t & \omega' \end{pmatrix}\right] \\ \tilde{\underline{\omega}} &= \left[\begin{pmatrix} \omega & 0 \\ 0 & \Omega \end{pmatrix}, \begin{pmatrix} \Omega' & 0 \\ 0 & \omega' \end{pmatrix}\right] \\ \tilde{\tilde{\underline{\omega}}} &= \left[\begin{pmatrix} \omega & 0 \\ 0 & C \end{pmatrix}, \begin{pmatrix} C' & 0 \\ 0 & \omega' \end{pmatrix}\right]\end{aligned}$$

proof: Let $E_{\bar{a}} = (E_a, E_{a'})$ be the normed eigenvectors of I, *i.e.*, $IE_a = E_a, IE_{a'} = -E_{a'}$, then we get $\underline{\omega}$ from the definition $\nabla_{\bar{a}} E_{\bar{b}} =: \underline{\omega}_{\bar{a}\bar{b}}{}^{\bar{c}} E_{\bar{c}}$ and the definition of the deformation tensor $H_{ab}{}^{c'} := \mathcal{P}'\underline{\nabla}_a E_b = \omega_{ab}{}^{c'} E_{c'}$. We have furthermore denoted the normal connections by Ω, *i.e.*, $\underline{\omega}_{ab'}{}^{c'} =: \Omega_{ab'}{}^{c'}$. Now from the relation $A(X,Y) = \frac{1}{2}(I\underline{\nabla}_X IY - \underline{\nabla}_X Y)$ we get in the same basis

$$A_{\bar{a}\bar{b}}{}^{\bar{c}} = \left[\begin{pmatrix} 0 & -H_{ab}{}^{c'} \\ -H_{ab'}{}^{c} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -H'{}_{a'b'}{}^{c} \\ -H'{}_{a'b}{}^{c'} & 0 \end{pmatrix}\right]$$

so $\tilde{\underline{\omega}}$ follows. If we write $B(X,Y) = \frac{1}{4}(\underline{\nabla}_{IY} Ix - I\underline{\nabla}_{IY} X + I\underline{\nabla}_Y IX - \underline{\nabla}_Y X)$ we similarly get

$$B_{\bar{a}\bar{b}}{}^{\bar{c}} = \left[\begin{pmatrix} 0 & 0 \\ 0 & -H'{}_{b'a}{}^{c'} \end{pmatrix}, \begin{pmatrix} -H_{ba'}{}^{c} & 0 \\ 0 & 0 \end{pmatrix}\right].$$

Finally, from the torsion equation we have $0 = \omega_{ab'}{}^{c'} - \omega_{b'a}{}^{c'} - C_{ab'}{}^{c'} \Rightarrow \Omega_{ab'}{}^{c'} - H'_{b'a}{}^{c'} = C_{ab'}{}^{c'}$. ■

4.3 Almost product manifolds, the classification

In this section we will present the different classes of riemannian almost product structures, which will be shown to follow from the different classes of deformation tensors of section (3). These different classes are primarily split into three different types, basically refering to the three cases when either both distributions associated with an almost product structure are integrable, only one is, or the last type where none is integrable. In the first case the manifold is doubly foliated, in the second singly, and in the third not foliated at all, of course with respect to the almost product structure.

4.3.1 The types defined by the Nijenhuis tensor

To begin with we will see that there are relations between the Nijenhuis tensor and the two new connections introduced in the preceding subsection. These relations are characterized by only involving the torsion parts of the two connections.

Proposition 4.38
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define an riemannian almost product structure, let N_I denote the Nijenhuis tensor of I and $\tilde{\underline{\nabla}}$ the adapted connection defined in 4.31, then we have the following relation

$$\frac{1}{2}N_I(X,Y) = \tilde{T}(X,Y) + \tilde{T}(IX,IY)$$

proof: By definition 4.31 we get

$$\begin{aligned}\tilde{T}(X,Y)+\tilde{T}(IX,IY) &= \tilde{\underline{\nabla}}_X Y - \tilde{\underline{\nabla}}_Y X - [X,Y] + \tilde{\underline{\nabla}}_{IX} IY - \tilde{\underline{\nabla}}_{IY} IX - [IX,IY] = \\ &= \frac{1}{2}(\underline{\nabla}_X Y + I\underline{\nabla}_X IY - \underline{\nabla}_Y X - I\underline{\nabla}_Y IX) - [X,Y] + \\ &\quad + \frac{1}{2}(\underline{\nabla}_{IX} IY + I\underline{\nabla}_{IX} Y - \underline{\nabla}_{IY} IX - I\underline{\nabla}_{IY} X) - [IX,IY] = \\ &= \frac{1}{2}(I\underline{\nabla}_X IY - I\underline{\nabla}_Y IX + I\underline{\nabla}_{IX} Y - I\underline{\nabla}_{IY} X - I^2[X,Y] - [IX,IY]) = \\ &= \frac{1}{2}(I[X,IY] + I[IX,Y] - I^2[X,Y] - [IX,IY]) = \\ &= \frac{1}{2}N_I(X,Y).\end{aligned}$$

■

In the case of the Vidal connection, which in the classification scheme will be more important to us, we have an even stronger relation.

Proposition 4.39
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define an riemannian almost product structure, let N_I denote the Nijenhuis tensor of I and $\tilde{\tilde{\underline{\nabla}}}$ denote the Vidal connection defined in 4.32, then we have the relation

$$\frac{1}{4}N_I(X,Y) = \tilde{\tilde{T}}(X,Y)$$

proof: Similar to the proof in 4.38. ■

We see that the torsion of the Vidal connection has a one-to-one correspondence with the Nijenhuis tensor while in the case of the adapted connection the torsion tensor contains the Nijenhuis tensor but also some additional terms. These terms are in fact the entire deformation tensor, so the adapted connection is not very suitable for our study of different riemannian almost product structures. Now putting all information involving the Nijenhuis tensor together, we get the following theorem.

Theorem 4.40
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define a riemannian almost product structure, let $\mathcal{D}, \mathcal{D}'$ be the associated distributions and let L, L' be the skew tensors of the deformation, then the first type of almost product structure corresponding to a doubly foliated manifold can be seen by the following equivalent statements,

$$\begin{aligned} (i) &\qquad N_I = 0, \\ (ii) &\qquad L = 0, \quad L' = 0, \\ (iii) &\qquad \underline{\tilde{\tilde{\nabla}}} \quad \text{is torsionless}, \\ (iv) &\qquad \mathcal{D}, \mathcal{D}' \quad \text{are integrable.} \end{aligned}$$

proof: From propositions 4.27, 4.39 and 3.27. ■

So we see, in the case where the endomorphism I denotes a riemannian almost product structure, that the Nijenhuis tensor contains two parts L, L', measuring the failure of integrability in the two complementary distributions $\mathcal{D}, \mathcal{D}'$ respectively. We also see that an equivalent treatment is to look at the torsion of the Vidal connection which also measures the failure of integrability of the two complementary distributions associated with I. Here we manifestly see the splitting of riemannian almost product structures into three different types characterized by different conditions on the Nijenhuis tensor.

Proposition 4.41
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ denote a riemannian almost product structure, let N_I denote the Nijenhuis tensor associated with I, then N_I characterizes three different types by

$$\begin{aligned} N_I(X,Y) &= 0 &&\qquad \textbf{doubly foliated} \\ IN_I(X,Y) &= \pm N_I(X,Y) &&\qquad \textbf{singly foliated} \\ IN_I(X,Y) &\neq \pm N_I(X,Y) &&\qquad \textit{no foliation.} \end{aligned}$$

We will see some examples involving the two types of foliated almost product structures later, but we will first examine what extra structure these two types give us. We will start by introducing two new differential operators associated with an almost product structure I.

Definition 4.42
Let I be an almost product structure on a manifold $\underline{\mathscr{M}}$ with exterior derivative $\underline{d}$. Let furthermore $\underline{d}_I$ denote the exterior derivative associated with I and define two new differential operators by

$$\begin{aligned} d &:= \frac{1}{2}(\underline{d} + \underline{d}_I) \\ d' &:= \frac{1}{2}(\underline{d} - \underline{d}_I) \end{aligned}$$

An equivalent definition is by the two projection operators defined by the endomorphism I, $\mathcal{P} := \frac{1}{2}(1 + I)$ and $\mathcal{P}' := \frac{1}{2}(1 - I)$, then the operators are simply $d \equiv \underline{d}_{\mathcal{P}}$ and $d' \equiv \underline{d}_{\mathcal{P}'}$.

These differential operators will be of utmost importance in the case where we have a vanishing Nijenhuis tensor.

Proposition 4.43
Let I be an almost product structure on a manifold $\underline{\mathscr{M}}$, and N_I the Nijenhuis tensor associated with I, then if $N_I = 0$ the new differential operators defined in 4.42 will be nilpotent and thus coboundary operators. The exterior algebra will become doubly graded with respect to these new coboundary operators.

proof: We know from lemma 4.26 that $N_I = 0 \Rightarrow N_{\mathcal{P}} = N_{\mathcal{P}'} = 0$ why both d and d' are nilpotent. They are thus coboundary operators. Because of the doubly foliated structure we know that they can be expressed locally by $d = dx^m \partial_m$ and $d' = dy^{m'} \partial_{m'}$. ∎

We see that if the Nijenhuis tensor vanishes the new differential operators are in fact coboundary operators and the exterior algebra becomes doubly graded under these two coboundary operators.

Definition 4.44
Let the triple $(\underline{\mathscr{M}}, \underline{g}, I)$ define a riemannian almost product structure, let $N_I = 0$ and denote the set of doubly graded forms on $\underline{\mathscr{M}}$ by $\Omega^{p,q} = \Omega^{p,q}(\underline{\mathscr{M}})$ characterized by

$$\omega = \frac{1}{(p+q)!} \omega_{m_1 \dots m_p m'_1 \dots m'_q}(x,y) dx^{m_1} \wedge \dots dx^{m_p} \wedge dy^{m'_1} \wedge \dots dy^{m'_q}$$

where $\omega \in \Omega^{p,q}$. We thus see that the new coboundary operators defined in 4.42 have the following characteristics:

$$\begin{aligned} d :& \Omega \longmapsto \Omega, \qquad \Omega^{p,q} \longmapsto \Omega^{p+1,q} \\ d' :& \Omega \longmapsto \Omega, \qquad \Omega^{p,q} \longmapsto \Omega^{p,q+1} \end{aligned}$$

and that the graded algebra of differential forms now becomes doubly graded. The coboundary operators trivially satisfy the relations

$$\begin{aligned} d^2 &= 0, \\ d'^2 &= 0, \\ dd' + d'd &= 0. \end{aligned}$$

We see that it is in complete analogy to the case of an almost complex structure, where the vanishing of the Nijenhuis tensor tells us that we have a complex manifold which gives us a doubly graded exterior algebra under holomorphic and anti-holomorphic coordinates. In this case though we have a splitting which looks topologically like a product, taken into account that the almost product structure I defines an Ehresmann foliation. This requirement is just that taking any curve in a leaf of the foliation $\mathcal{F}$ and lifting it to another leaf by following only normal directions, the quotient of their respective lengths shall exist. This is the same as saying that the curves do not shrink to zero or blow up to infinity as we lift them by normal curves. There do exist foliations which have these types of singularities, also called Reeb components, see [23]. In the case of a riemannian almost product structure defining a Ehresmann foliation this amounts to saying that we are assured that the induced metrics on the two complementary distributions exist and are non-singular. So letting I define an Ehresmann foliation with vanishing Nijenhuis tensor it follows that the universal covering space splits to a topological product, $\underline{\tilde{\mathscr{M}}} = \tilde{\mathscr{M}} \times \tilde{\mathscr{M}}'$, where the tilde denotes the universal covering space and the product is in the topological sense, see [4, 5]. In an analogous way to the complex case, we also get a splitting of the cohomology groups under these two new coboundary operators and the double gradation.

Theorem 4.45
Let $\underline{\mathscr{M}}$ be a manifold, let I be an almost product structure on $\underline{\mathscr{M}}$ that defines an Ehresmann connection, then the vanishing of the Nijenhuis tensor implies that the de Rahm cohomology groups on $\underline{\mathscr{M}}$ splits like

$$H^p(\mathbb{R}) = \bigoplus_{p=k+l} H^{k,l}(\mathbb{R})$$

proof: See [4] ■

In the case of theorem 4.45 we see that the basic cohomology groups map isomorphically into these doubly graded cohomology groups. We have in this case $H^p_{B_{\mathcal{F}}} = H^{0,p}$. Let us finally list some local properties of the different tensors involved. We put them in a proposition but the proof will be immediate.

Proposition 4.46
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define a riemannian almost product structure, then we have three basic types defined by the Nijenhuis tensor. We will see how the local structure of the tensors involved look.

(i) **doubly foliated** $\Leftrightarrow N_I = 0$.
In this case, where the Nijenhuis tensor of the almost product structure vanishes, we have a doubly graded tensor algebra. We can therefore write the oriented vielbeins on the form

$$E_a = E_a{}^m \partial_m, \qquad E_{a'} = E_{a'}{}^{m'} \partial_{m'},$$
$$E^a = dx^m E_m{}^a, \qquad E^{a'} = dy^{m'} E_{m'}{}^{a'},$$

where of course $E_a{}^m, E_{a'}{}^{m'}, E_m{}^a, E_{m'}{}^{a'}$ are functions on $\underline{\mathscr{M}}$ satisfying $E_a{}^m E_m{}^b = \delta_a{}^b, E_m{}^a E_a{}^n = \delta_m{}^n, E_{a'}{}^{m'} E_{m'}{}^{b'} = \delta_{a'}{}^{b'}, E_{m'}{}^{a'} E_{a'}{}^{n'} = \delta_{m'}{}^{n'}$.

The metric takes the form

$$\begin{aligned} \underline{g} &= \eta_{ab}E^aE^b + \eta_{a'b'}E^{a'}E^{b'} = \\ &= g_{mn}(x,y)dx^m dx^n + g'_{m'n'}(x,y)dy^{m'}dy^{n'}. \end{aligned}$$

where we have used η to stress that we can have any signature of the metric. We also see that the almost product structure takes the simple form

$$\begin{aligned} I &= E^aE_a - E^{a'}E_{a'} = \\ &= dx^m\partial_m - dy^{m'}\partial_{m'}, \end{aligned}$$

so the Nijenhuis tensor vanishes. We also find the two associated boundary operators to be

$$\begin{aligned} d &= dx^m\frac{\partial}{\partial x^m}, \\ d' &= dy^{m'}\frac{\partial}{\partial y^{m'}}. \end{aligned}$$

(ii) **singly foliated** $\Leftrightarrow IN_I = \pm N_I$.
In this case only one set of vielbeins associated to I defines a foliation which we will take to be the unprimed set, i.e., $IN_I = N_I$. The vielbeins can now be expressed in the form

$$\begin{aligned} E_a &= E_a{}^m\partial_m, & E_{a'} &= E_{a'}{}^{m'}(\partial_{m'} + A_{m'}{}^m\partial_m), \\ E^a &= (dx^m - dy^{m'}A_{m'}{}^m)E_m{}^a, & E^{a'} &= dy^{m'}E_{m'}{}^{a'}, \end{aligned}$$

now additionally $A_{m'}{}^m$ are functions on $\underline{\mathscr{M}}$. It is convenient to introduce objects $D_{m'} := \partial_{m'} + A_{m'}{}^m\partial_m$ and $\Pi^m := dx^m - dy^{m'}A_{m'}{}^m$ such that the vielbeins instead can be written in the simpler form

$$\begin{aligned} E_a &= E_a{}^m\partial_m, & E_{a'} &= E_{a'}{}^{m'}D_{m'} \\ E^a &= \Pi^m E_m{}^a, & E^{a'} &= dy^{m'}E_{m'}{}^{a'}. \end{aligned}$$

Now there will be no surprise that $D_{m'}$ in fact will be the covariant derivative in the example of foliations in principle bundles that we will see later. The metric takes the form

$$\begin{aligned} \underline{g} &= \eta_{ab}E^aE^b + \eta_{a'b'}E^{a'}E^{b'} = \\ &= g_{mn}(x,y)\Pi^m\Pi^n + g'_{m'n'}(x,y)dy^{m'}dy^{n'}. \end{aligned}$$

where the non-integrability of the prime distribution makes itself manifest through the differentials Π^m. We find the almost product structure to be of the form

$$\begin{aligned} I &= E^aE_a - E^{a'}E_{a'} = \\ &= \Pi^m\partial_m - dy^{m'}D_{m'} \end{aligned}$$

why the associated Nijenhuis tensor fails to vanish but instead reads

$$\begin{aligned} -N_{I\,m'n'} &= \mathcal{P}[D_{m'}, D_{n'}] = \\ &= \partial_{m'}A_{n'} - \partial_{n'}A_{m'} + [A_{m'}, A_{n'}]. \end{aligned}$$

It thus measures at what extent the prime distribution fails to be integrable. The two associated differential operators become

$$\begin{aligned} d &= \Pi^m \partial_m, \\ d' &= dy^{m'} D_{m'}. \end{aligned}$$

Let I define an Ehresmann connection, and thus a fibration. Denote it by $0 \to \mathscr{M}_{\mathcal{F}} \to \underline{\mathscr{M}} \to \mathscr{M}'_{\mathcal{D}} \to 0$, where $\mathscr{M}_{\mathcal{F}}$ is the leaf of the foliation and $\mathscr{M}'_{\mathcal{D}} = \underline{\mathscr{M}}/\mathscr{M}_{\mathcal{F}}$ is the leafspace. Let further σ be a section of the leafspace in $\underline{\mathscr{M}}$, then the covariant derivative on the leafspace is simply $D' = \sigma^* \underline{d} = \sigma^* d' = d'|_\sigma$. The curvature of this covariant derivative is nothing but the Nijenhuis tensor.

(iii) **no** *foliation $\Leftrightarrow$ no condition.*
In this case we have no foliation and thus the sets of vielbeins will none be of a simple form but both needs to be expressed in terms of both ∂_m and $\partial_{m'}$. This case will be of no interest to us as we practically get no extra structure of importance.

4.3.2 The classes defined by the Jordan tensor

We will here proceed to get the extra structure to a riemannian almost product structure by looking at the Jordan tensor. If we put everything we have regarding the Jordan tensor together we end up with the theorem.

Theorem 4.47
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define a riemannian almost product structure, let $\mathcal{D}, \mathcal{D}'$ be the associated distributions and let L, L' be the skew tensors of the deformation, then the first type of almost product structure corresponding to a doubly foliated manifold can be seen by the following equivalent statements,

$$\begin{aligned} &(i) && M_I = 0, \\ &(ii) && K = 0, \quad K' = 0, \\ &(iii) && \underline{\tilde{\tilde{\nabla}}} \quad \text{is metric}, \\ &(iv) && \mathcal{D}, \mathcal{D}' \quad \text{are geodesic}. \end{aligned}$$

proof: (i), (ii) and (iv) is clear from proposition 4.30 and definition 3.29, now we need to prove (iii), that is we need to prove that

$$\begin{aligned} \underline{g}(B(X,Y),Z) + \underline{g}(Y, B(X,Z)) = \\ &= \frac{1}{4}(\underline{g}((I\underline{\nabla}_Y I + \underline{\nabla}_{IY})(X), Z) + \underline{g}(Y, (I\underline{\nabla}_Z I + \underline{\nabla}_{IZ})(X))) = \\ &= -\frac{1}{4}(\underline{g}(X, (I\underline{\nabla}_Y I - \underline{\nabla}_{IY})(Z)) + \underline{g}((I\underline{\nabla}_Z I - \underline{\nabla}_{IZ})(Y), X)) = \\ &= -\frac{1}{4}\underline{g}(X, H_I(Y,Z) + H_I(Z,Y)) = \\ &= -\frac{1}{8}\underline{g}(X, M_I(Y,Z)), \end{aligned}$$

and the equivalence is clear. ■

We see that we have a similar structure as in the case for the Nijenhuis tensor. Now the Jordan tensor measures whether the two complementary distributions are geodesic or not while the Nijenhuis tensor measured whether they were integrable. We will soon proceed to split the Jordan tensor further and look at the traceless and trace parts of it to divide up the classes further, but first we will look at the special case when the almost product structure I is covariantly constant, as we will see a typical analogue to the complex case.

Theorem 4.48
Let the 3-tuple ($\underline{\mathscr{M}}$, $\underline{g}$, I) define a riemannian almost product structure with Levi–Civita connection $\underline{\nabla}$. Let $\tilde{\underline{\nabla}}$ denote the adapted connection and $\tilde{\tilde{\underline{\nabla}}}$ the Vidal connection then the following equivalence holds

$$\underline{\nabla} I = 0 \quad \Leftrightarrow \quad \tilde{\tilde{\underline{\nabla}}} = \tilde{\underline{\nabla}} = \underline{\nabla}$$

proof: Immediate from definitions 4.31 and 4.32. ■

The first obvious consequence of this is that the Nijenhuis tensor vanishes. Note that in the case of a Kähler manifold we know that as the almost complex structure, J, is covariantly constant, *i.e.*, $\underline{\nabla} J = 0$, J is also integrable.

Corollary 4.49
Let the 3-tuple ($\underline{\mathscr{M}}$, $\underline{g}$, I) define a riemannian almost product structure and let $\underline{\nabla}$ be the Levi–Civita connection then

$$\underline{\nabla} I = 0 \quad \Rightarrow \quad N_I = 0$$

proof: From 4.48 and proposition 4.37. ■

We also know from the complex case that Kähler implies reduction of the holonomy groups so it is no surprise that we find it in the case of a covariantly constant almost product structure to.

Corollary 4.50
Let the 3-tuple ($\underline{\mathscr{M}}$, $\underline{g}$, I) define a riemannian almost product structure. If the adapted connection and the Vidal connection are Levi–Civita then the holonomy group splits as

$$O(m) = O(k) \times O(k')$$

which follows from the splitting of the Lie algebra of the connection

$$\mathfrak{o}(m) = \mathfrak{o}(k) \oplus \mathfrak{o}(k')$$

proof: From proposition 4.37. ■

In the case of a Kähler manifold we know that the holonomy group reduces to $U(m) \subset O(2m)$ where $U(m)$ is a subgroup of the generic holonomy group $O(2m)$ of a $2m-$dimensional manifold, while in the case of the covariantly constant almost product structure we get the subgroup $O(k) \times O(m-k)$ instead of the generic holonomy group $O(m)$. From what we have seen the case of a covariantly constant almost product structure tells us that the universal covering space in fact is a product manifold.

Theorem 4.51
Let the 3-tuple ($\underline{\mathscr{M}}$, $\underline{g}$, I) define a riemannian almost product structure. If now the Vidal connection $\tilde{\tilde{\underline{\nabla}}}$ is Levi–Civita (i.e., metric and torsionless), then $\underline{\tilde{\mathscr{M}}}$, the universal covering space of $\underline{\mathscr{M}}$, is a product manifold.

proof: From proposition 4.47 plus the fact that it is a topological product from the vanishing of the Nijenhuis tensor. ■

Now we will continue to split the Jordan tensor into its traceless and trace parts to get the four classes of distributions in the geometric sense, namely geodesic, umbilic, minimal, and the last with no condition. So as we saw in definition 3.29 we have eight different classes of a distribution and now in the case of a almost product structure which leaves us with two complementary distributions we thus get 64 different classes. Now it is immediate that it does not matter which we call complementary of the two distributions so we have in fact only 36 different classes, see [22].

Proposition 4.52
Let the triplet ($\underline{\mathscr{M}}, \underline{g}, I$) be an riemannian almost product structure. We then

have the following 36 different classes

Classes	L	W	κ	L'	W'	κ'	Name
(GF,GF)	x	x	x	x	x	x	*Local product*
(GF,UF)	x	x	x	x	x		*Twisted product*
(GF,MF)	x	x	x	x		x	
(GF,F)	x	x	x	x			
(UF,UF)	x	x		x	x		*Double twisted product*
(UF,MF)	x	x		x		x	
(UF,F)	x	x		x			
(MF,MF)	x		x	x		x	
(MF,F)	x		x	x			
(F,F)	x			x			
(GF,GD)	x	x	x		x	x	*Riemannian foliation*
(UF,GD)	x	x			x	x	*Riemannian foliation*
(MF,GD)	x		x		x	x	*Riemannian foliation*
(F,GD)	x				x	x	*Riemannian foliation*
(GF,UD)	x	x	x		x		
(UF,UD)	x	x			x		
(MF,UD)	x		x		x		
(F,UD)	x				x		
(GF,MD)	x	x	x			x	
(UF,MD)	x	x				x	
(MF,MD)	x		x			x	
(F,MD)	x					x	
(GF,D)	x	x	x				
(UF,D)	x	x					
(MF,D)	x		x				
(F,D)	x						
(GD,GD)		x	x		x	x	
(GD,UD)		x	x		x		
(GD,MD)		x	x			x	
(GD,D)		x	x				
(UD,UD)		x			x		
(UD,MD)		x				x	
(UD,D)		x					
(MD,MD)			x			x	
(MD,D)			x				
(D,D)							

The structure added to the various tensors in some of the different classes will be put as a proposition. Again the proof is immediate.

Proposition 4.53
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define a riemannian almost product structure, let the Nijenhuis tensor define the three different types of almost product structures as in 4.46, then we have additionally the following examples of classes in various types

(i) **doubly foliated** $\Leftrightarrow N_I = 0$.
In this type we have 10 different classes as seen in proposition 4.52. We will take a closer look at the local structure of the metric in some of these classes.

$$\begin{aligned}
(GF, GF): \quad & \underline{g} = g_{mn}(x)dx^m dx^n + g'_{m'n'}(y)dy^{m'}dy^{n'} \\
(GF, UF): \quad & \underline{g} = g_{mn}(x)dx^m dx^n + \lambda'(x,y)g'_{m'n'}(y)dy^{m'}dy^{n'} \\
(UF, UF): \quad & \underline{g} = \lambda(x,y)g_{mn}(x)dx^m dx^n + \lambda'(x,y)g'_{m'n'}(y)dy^{m'}dy^{n'}
\end{aligned}$$

These are the classes referred to as **local product**, **twisted product** *and* **doubly twisted product** *respectively. In all these cases we know that $W = 0, W' = 0$, but in the cases where we have twisted products the mean curvature does not vanish. In fact we have*

$$\begin{aligned}
(GF, UF): \quad & \kappa = 0 & , \kappa' = -\frac{k'}{2}\lambda'^{-1}d\lambda'(x,y), \\
(UF, UF): \quad & \kappa = -\frac{k}{2}\lambda^{-1}d'\lambda(x,y), & \kappa' = -\frac{k'}{2}\lambda'^{-1}d\lambda'(x,y).
\end{aligned}$$

These two classes are called **warped product** *and* **doubly warped product** *respectively when the conformal factors λ, λ' only depends on the coordinates $y^{m'}, x^m$ respectively. This gives us*

$$\begin{aligned}
(GF, UF): \quad & \underline{g} = g_{mn}(x)dx^m dx^n + \lambda'(x)g'_{m'n'}(y)dy^{m'}dy^{n'}, \\
(UF, UF): \quad & \underline{g} = \lambda(y)g_{mn}(x)dx^m dx^n + \lambda'(x)g'_{m'n'}(y)dy^{m'}dy^{n'},
\end{aligned}$$

and the mean curvatures now become basic 1-forms and take the form

$$\begin{aligned}
(GF, UF): \quad & \kappa = 0, & \kappa' = -\frac{k'}{2}\lambda'^{-1}d\lambda'(x), \\
(UF, UF): \quad & \kappa = -\frac{k}{2}\lambda^{-1}d'\lambda(y), & \kappa' = -\frac{k'}{2}\lambda'^{-1}d\lambda'(x).
\end{aligned}$$

We will later see that this case is the interesting case of the present brane solutions in M-theory.

(ii) **singly foliated** $\Leftrightarrow IN_I = \pm N_I$.
As in 4.46 we will look at the case where $IN_I = N_I$ and find that some structure is inherited from the doubly foliated case. Of this type we have 16 classes of which we will list some examples.

$$\begin{aligned}
(F, GD): \quad & \underline{g} = g_{mn}(x,y)\Pi^m\Pi^n + g'_{m'n'}(y)dy^{m'}dy^{n'}, \\
(F, UD): \quad & \underline{g} = g_{mn}(x,y)\Pi^m\Pi^n + \lambda(x,y)g'_{m'n'}(y)dy^{m'}dy^{n'}.
\end{aligned}$$

The first of these are called **riemannian foliations** *which are characterized by the complementary distribution being geodesic. Or to put it in the classification scheme, $(*F, GD)$. In this case the metric $\underline{g}$ is said* *to be bundle-like and we have $\mathscr{L}_X g' = 0$. So the vectors of the foliation are isometries of the complementary metric, g'. In the second case we see that we get a non-vanishing mean curvature form for the complementary distribution*

$$\kappa'_m = -\frac{k'}{2}\lambda'^{-1}\partial_m\lambda'.$$

If we now let the foliation be geodesic we get

$$\begin{aligned}(GF, GD): \quad & \underline{g} = g_{mn}(x)\Pi^m\Pi^n + g'_{m'n'}(y)dy^{m'}dy^{n'},\\ (GF, UD): \quad & \underline{g} = g_{mn}(x)\Pi^m\Pi^n + \lambda(x,y)g'_{m'n'}(y)dy^{m'}dy^{n'},\end{aligned}$$

where additionally $A_{m'}{}^a = A_{m'}{}^a(y), C_{a(bc)} = 0$. This is the case where the leaves of the foliation are a Lie group for instance. We will later see that a principal bundle lies in the first of these classes. Furthermore, we could let the foliation become umbilic and get an analogue of the type one case, but we will restrict to the case where the metric takes the form

$$(UF, GD): \quad \underline{g} = \lambda(y)g_{mn}(x)\Pi^m\Pi^n + g'_{m'n'}(y)dy^{m'}dy^{n'}.$$

We will see that in Kaluza–Klein theory this is the case when introducing the scalar field $\lambda = e^{-2\phi}$ which measures the radius of the gauge group. Here we get

$$\kappa_{m'} = -\frac{k}{2}\lambda^{-1}\partial_{m'}\lambda = k\partial_{m'}\phi.$$

(iii) **no** *foliation $\Leftrightarrow$ no condition. As this case is rather uninteresting we will only say that in the case of a geodesic distribution the extrinsic curvature vanishes which can be viewed in the form*

$$(GD): \quad K_{abc'} = C_{c'(ab)} = 0.$$

We can now use this formalism to study the structure of for instance the brane solutions of M-theory. The following example tells how the different tensors look and what they say.

Example 4.54

M2-brane

In the M2-brane case we have the solution to the equations of motion for the metric in the form [3, 10]

$$\underline{g} = \Delta^{-\frac{2}{3}}(y)\eta_{mn}dx^m dx^n + \Delta^{\frac{1}{3}}(y)\delta_{m'n'}dy^{m'}dy^{n'}$$

where

$$\Delta(y) = 1 + (\frac{a}{\rho(y)})^6, \qquad \rho(y) = \sqrt{\delta_{m'n'}y^{m'}y^{n'}}.$$

and $\rho = 0$ is the horizon of the brane not the core. We find the corresponding vielbeins

$$\begin{aligned} E_a =& \Delta^{-\frac{1}{3}} \delta_a{}^m \partial_m, \qquad & E_{a'} =& \Delta^{\frac{1}{6}} \delta_{a'}{}^{m'} \partial_{m'}, \\ E^a =& \Delta^{\frac{1}{3}} dx^m \delta_m{}^a, \qquad & E^{a'} =& \Delta^{-\frac{1}{6}} dy^{m'} \delta_{m'}{}^{a'}, \end{aligned}$$

which we use to derive the almost product structure which splits the tangent bundle accordingly

$$\begin{aligned} I &= E^a E_a - E^{a'} E_{a'} = \\ &= dx^m \partial_m - dy^{m'} \partial_{m'}. \end{aligned}$$

By definition, $I^2 = 1$, and we see that not only the brane is integrable but also the complementary distribution associated with I. Accordingly the Nijenhuis tensor vanishes,

$$N_I = 0.$$

So we see that this typical solution is a doubly foliated manifold in the class (UF,GF) and additionally it is spherical why the metric is nothing but a warped-product. We get the mean curvature

$$\kappa = \Delta^{-1} d' \Delta,$$

and as W vanishes we see that generating translations radially from the brane by the vector $\partial/\partial\rho$ we generate conformal transformations on the brane.

M5-brane

In the M5-brane case we will look at two types of solutions, the first one of which is the ordinary with no field excitations on the brane [14, 10], the second where we have excited the anti self-dual tensor field on the brane found recently [7]. The first solution of the metric looks like

$$\underline{g} = \Delta^{-\frac{1}{3}}(y) \eta_{mn} dx^m dx^n + \Delta^{\frac{2}{3}}(y) \delta_{m'n'} dy^{m'} dy^{n'}$$

where

$$\Delta(y) = 1 + (\frac{a}{\rho(y)})^3, \qquad \rho(y) = \sqrt{\delta_{m'n'} y^{m'} y^{n'}}.$$

The metric and thus the vielbeins look very similar to the M2-brane case

$$\begin{aligned} E_a =& \Delta^{-\frac{1}{6}} \delta_a{}^m \partial_m, \qquad & E_{a'} =& \Delta^{\frac{1}{3}} \delta_{a'}{}^{m'} \partial_{m'}, \\ E^a =& \Delta^{\frac{1}{6}} dx^m \delta_m{}^a, \qquad & E^{a'} =& \Delta^{-\frac{1}{3}} dy^{m'} \delta_{m'}{}^{a'}, \end{aligned}$$

Again we get $I = dx^m \partial_m - dy^{m'} \partial_{m'}$ and $N_I = 0$ but now of course with a different number of x and y directions. So we see also in this case that we have a doubly foliated solution of the type (UF, GF). Notable is that this implies that the antisymmetric tensor fields which where a basic form now lies in the graded cohomology group

$$H \in H^{0,4}.$$

The mean curvature is the same as in the M2-brane case

$$\kappa = \Delta^{-1} d' \Delta,$$

but now with a different function Δ of course. In the other solution with the tensor field excited the metric looks like

$$\underline{g} = (\Delta_+\Delta_-)^{-1/6}\left[\left(\frac{\Delta_+}{\Delta_-}\right)^{1/2} dx_-^2 + \left(\frac{\Delta_-}{\Delta_+}\right)^{1/2} dx_+^2\right] + (\Delta_+\Delta_-)^{1/3} dy^2$$

where $\Delta_+ = \Delta + \nu, \Delta_- = \Delta - \nu$ and Δ is as before. Here we have yet another almost product structure lying in the brane denoted q which squares to one. In this case we find that W does not vanish anymore but will in fact be

$$^\flat W = \frac{1}{4}(\frac{1}{\Delta_+} - \frac{1}{\Delta_-})^\flat q d'\Delta$$

and the mean curvature will read

$$\kappa = \frac{1}{2}(\frac{1}{\Delta_+} + \frac{1}{\Delta_-})d'\Delta$$

so we see that we have a solution in the class (F, GF). Interesting to see is that the new almost product structure in the brane, q, defines three new preferred directions which in some sense can be seen as a membrane, see [7] for further information. Here we just state the utmost importance in studying several almost product structures on a manifold as these would result in multi brane configurations. Interesting would be to see what conditions would be implied on these almost product structure if we furthermore require that these associated brane configurations would solve the equations of motions.

Now as we said earlier the structure of the Nijenhuis tensor took such a form that we suspected that it measured the curvature of fibrations. It will be clear from the next example, where we look at principal bundles, that this is truly a fact. We will also see that Nijenhuis tensor measures the field strength in Kaluza–Klein theories.

Example 4.55

Let $P(\mathscr{M}, G)$ be a principal bundle with base space $\mathscr{M}$ and fiber G. Let furthermore $\underline{d}$ denote the exterior derivative in the total space, T_a denote the generators of the Lie algebra $\mathfrak{g}$ associated to the Lie group G, fulfilling the algebra $[T_a, T_b] = f_{ab}{}^c T_c$ and normalized like $\mathrm{tr}(T_aT_b) = \delta_{ab}$. The vielbeins can be expressed as

$$\begin{aligned} E^aT_a &= g^{-1}\underline{d}g + g^{-1}Ag, \qquad & E^{a'} &= dy^{m'} E_{m'}{}^{a'}, \\ E_a &= R_a, & E_{a'} &= E_{a'}{}^{m'}(\partial_{m'} - A_{m'}{}^a R_a) \end{aligned}$$

where $g^{-1}\underline{d}g(L_a) = T_a, g^{-1}\underline{d}g(R_a) = Ad_{g^{-1}}T_a$ and R_a and L_a are the right and left invariant vector fields on G respectively. Here we have done the split $T_uP = V_uP \oplus H_uP$ where the vertical subbundle is spanned by E_a and the horizontal by $E_{a'}$. We define the connection $\omega := E^aT_a$ [13, 17] and write the Lie algebra valued curvature form as

$$\Omega := \underline{d}\omega + \omega \wedge \omega$$

We now know [21] that taken two vectors $X, Y \in TP$ we get

$$\Omega(X, Y) = -\omega([X_H, Y_H]).$$

This tells us that if we expand the connection to $\tilde{\omega}(g,y) := l_{g*}\omega = E^a L_a$ *now giving* $\tilde{\omega}(X) = X_V$ *instead of pushing the vector back to the Lie algebra we see that, defining* $\tilde{\omega}' := 1_P - \tilde{w}$ *and* $I = \tilde{\omega} - \tilde{\omega}' = 2\tilde{\omega} - 1_P$*, we get from lemma 4.26*

$$\tilde{\Omega}(X,Y) = \frac{1}{4}N_I(X,Y)$$

or $\frac{1}{4}\omega(N_I(X,Y)) = \Omega(X,Y)$*. So we see that indeed the field strength of gauge theory is a special case of the Nijenhuis tensor only valid when the fiber is a gauge group. We can also easily see that this is nothing but a foliation of class (GF,GD). This due to the fact that by definition we have* $H_{ug}P = r_{g*}H_uP$ *which implies that* $A = A(y)$ *implying GF and* ${E_{a'}}^{m'} = {E_{a'}}^{m'}(y)$ *implying GD.*

In the case of algebraic gauge [18] we have the short exact sequence $0 \rightarrow A \rightarrow^i E \rightarrow^\pi B \rightarrow 0$ *where* A *is the fiber* E *the total space and* B *the base manifold all being algebras. We have a connection on* B *denoted* ρ *such that*

$$\rho : B \longmapsto E, \qquad \pi \circ \rho = 1_B.$$

Equivalently we can look at a connection in E *instead and denote it by* ω *now satisfying*

$$\omega : E \longmapsto A, \qquad \omega \circ i = 1_A.$$

We have the immediate relation between the two connections

$$\omega = 1_E - \rho \circ \pi$$

satisfying $\omega^2 = \omega$*. The curvature of these two connections are defined for* $X, Y \in \Lambda^1_B$ *and* $\underline{X}, \underline{Y} \in \Lambda^1_E$ *by*

$$\begin{aligned} F(X,Y) &:= \rho([X,Y]) - [\rho(X), \rho(Y)], \\ \Omega(\underline{X}, \underline{Y}) &:= F(\pi\underline{X}, \pi\underline{Y}). \end{aligned}$$

Now let $\omega' = 1_E - \omega = \rho \circ \pi$ *and* $I = \omega - \omega' = 1_E - 2\rho \circ \pi$ *then the curvature* Ω *is nothing but the Nijenhuis tensor or*

$$\Omega(\underline{X}, \underline{Y}) = \frac{1}{4}N_I(\underline{X}, \underline{Y})$$

which follows directly when expressing Ω *in terms of* ω*, see [18].*

Next example will be that of Kaluza-Klein theory.

Example 4.56

We can extend the above example to the case of Kaluza–Klein theory where the fiber needs not be the gauge group itself but rather having the gauge group as isometry group [12]. Again we split the space as a fibration looking first at the (GF, GD) *case. Let* $\underline{\mathscr{M}}$ *denote the total space and* $\mathscr{M}$ *the base space. Let furthermore* H *denote the fiber which and let* $(x^m, y^{m'})$ *be local coordinates such that* $\{\partial_m\}$ *spans the foliation,* H*, locally and thus write our adapted frames*

$$\hat{E}_a = {E_a}^m\partial_m, \qquad \hat{E}_{a'} = {E_{a'}}^{m'}\partial_{m'} + {A_{a'}}^m\partial_m$$

Let K_i *be the set of Killing vectors of the fiber fulfilling the algebra*

$$[K_i, K_j] = f_{ij}{}^k K_k$$

where of course $K_i = K_i{}^a(x)E_a$*, so we can express* $A = E^{a'}A^i_{a'}K_i$*. We also require that* $A^i = A^i(y)$ *and that* $f_{i(jk)}$*=0 (see proposition 4.53). The inverse vielbeins now read*

$$\hat{E}^a = E^a - A^i K_i^a, \qquad \hat{E}^{a'} = E^{a'},$$

from which we derive the metric of the total space

$$\begin{aligned} \underline{g} =& g + g' = \eta_{ab}\hat{E}^a\hat{E}^b + \eta_{a'b'}\hat{E}^{a'}\hat{E}^{b'} = \\ =& \eta_{ab}(E^a - A^i K_i^a)(E^b - A^j K_j^b) + \eta_{a'b'}E^{a'}E^{b'}. \end{aligned}$$

We also use the set of vielbeins to form the almost product structure which splits the space according to the fibration. As it is a fibration this almost product structure can be seen as an Ehresmann connection on $\underline{\mathscr{M}}$*.*

$$I = \hat{E}^a\hat{E}_a - \hat{E}^{a'}\hat{E}_{a'} = E^a E_a - E^{a'}E_{a'} - 2A^i K_i$$

Of course $I^2 = 1$*, and if* $X, Y \in \Lambda^1$ *we have from lemma 4.26*

$$-N_I(X,Y) = \mathcal{P}[\mathcal{P}'X, \mathcal{P}'Y].$$

Let $\hat{E}_{m'} = \partial_{m'} + A^i_{m'}K_i =: D_{m'}$ *and note that* $\mathcal{P}K_i = K_i$ *then we again see that the Nijenhuis tensor measures the curvature*

$$\begin{aligned} -N_{I\,m'n'} =& \mathcal{P}[\hat{E}_{m'}, \hat{E}_{n'}] = \\ =& \mathcal{P}[\partial_{m'} + A^i_{m'}K_i, \partial_{n'} + A^j_{n'}K_j] = \\ =& (\partial_{m'}A^i_{n'} - \partial_{n'}A^i_{m'} + f_{jk}{}^i A^j_{m'}A^k_{n'})K_i = \\ =& F^i_{m'n'}K_i. \end{aligned}$$

From this analysis it is clear that we can extend this fibration to any case in the classification scheme, i.e., $(*F, *D)$*, where* $*$ *means* G, U, M *or nothing. So for example we can extend the theory to the* (UF, GD) *case where we have added one additional factor which can conformally transform the fiber as we move along the base space. So by letting* $\phi = \phi(y)$ *be a scalar field, often referred to as the dilaton, and rescale the vielbein as*

$$\tilde{E}^a := e^{-\phi}\hat{E}^a$$

then the mean curvature becomes

$$\kappa_{m'} = -\frac{1}{2}k\lambda^{-1}\partial_{m'}\lambda = k\partial_{m'}\phi,$$

telling us that the fiber now is umbilic instead or that movement on the base space generates conformal transformations on the fiber. The dilaton can now be seen as measuring the radius of the fiber. We could of course go further by relaxing the conditions on the foliation and the complementary distribution further. Notable is that if we want to relax the condition on the fiber further we will break some of the isometries so the Killing vector algebra will reduce. We also note that any further relaxation of these kinds will not alter the Nijenhuis tensor and thus not the gauge field strength either.

5 Conclusions and outlook

We have demonstrated that we can express gauge theory, Kaluza–Klein theory and brane theory as special cases of almost product manifolds. We have also seen that the Nijenhuis tensor of certain almost product structures measures integrability which in gauge theories and Kaluza–Klein theories implies that the field strength measures the non-integrability of the complementary distribution to the foliation associated with the fiber (as we found them to be equal). Now there are lots of things that could be investigated further, one is how this almost product structure appears in the Clifford algebra. Another thing is to generalize all this to superspace, as we know from the embedding formalism [16, 2, 1] that a simple constraint gives us the right multiplets and brings us on shell. In a forthcoming paper [15], we will show how flat superspace can be seen in this formalism. But the most interesting continuation would of course be to find new solutions that are non trivial and maybe even show the existence of solutions in which the leafspace is non-commutative.

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Paper V

Curvature relations in almost product manifolds

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Abstract

New relations involving curvature components for the various connections appearing in the theory of almost product manifolds are given and the conformal behaviour of these connections are studied. New identities for the irreducible parts of the deformation tensor are derived. Some direct physical applications in Kaluza–Klein and gauge theory are discussed.

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1 Introduction

In modern day theoretical physics one often deals with additional dimensions besides the ordinary four space-time ones. These extra dimensions manifest themselves in different forms. In gauge theory they appear as the dimension of the gauge group, in M-theory and string theory, they are required for self-consistency. In Kaluza–Klein theory, the gauge theory is obtained by compactification over an internal manifold with an isometry group which equals the gauge group. The almost product structure concept makes possible a geometrical formulation which completely describe these theories, without performing the dimensional reduction. This leads to new insights in their geometrical properties which is unobtainable in the dimensionally reduced theories themselves. For instance in ref. [1] it was shown that the Nijenhuis tensor of a certain almost product structure measures the fieldstrength which in the geometrical language is a measure of the non-integrability of the base manifold of the principal bundle. In almost product manifolds, three different connections appear naturally. As is known the Gauss–Codazzi relations connect curvature components of these connections. In this paper a classification of the relations of all curvature components is given which yields a number of new identities. As a result it becomes manifest that the Vidal connection in a principal bundle, or Kaluza–Klein theory, reduces to the gauge-covariant derivative. Since a lot of recent work [2, 3] has been made concerning rotating branes which are solutions to various supergravity theories, it will here be stressed that the almost product manifold vievpoint would be the most geometrical approach to these problems. Direct relations for the Ricci tensors in terms of the characteristic deformation tensors of an almost product structure will be given. These will then be the most natural starting point when making ansätze for new solutions in the supergravity theories.

In the generic case, the Vidal connection will not be metric neither torsion free, and in section 2 we give a review of the theory of general connections. We refer to [4] for a more detailed treatise in this respect. In section 2 the properties of an arbitrary connection under conformal transformations are also reviewed. Section 3 gives a quick introduction to the basic connections and tensors involved with almost product manifolds. The naturally occuring connections, besides the Levi–Civita one, is the Vidal and adapted connections. All tensors formed from these connections are investigated in section 4. In that section several new identities are derived, some of which follows directly from the work in ref. [5], and the conformal properties are studied. In section 5 this is brought to full fruition when the Vidal connection is shown to be identical to the gauge-covariant derivative in gauge or Kaluza–Klein theory. Possible further developments in this area is discussed in section 6.

2 A review on general connections

This section consists of two parts, the first of which treats a general non-metric connection and its curvature relations together with the Bianchi identities. The second part deals with the induced transformation of an arbitrary connection under conformal transformations.

2.1 Non-metric connections

The most frequently used non-Levi-Civita connections are the ones in which the torsion content is non-zero. In the case of a Vidal connection, the connection will not in general be metric nor symmetric. Also in the case of embeddings one might encounter non-metric connections while studying cases with an auxilliary metric on the world-volume. In this subsection a thorough description of connections in the most general case is given. See also ref. [4]. To this end, the following two important tensor are defined as,

Definition 2.1
Let ∇ be a connection in a manifold $\mathscr{M}$ with non-degenerate metric g. Now define the torsion tensor, T, and the non-metricity tensor, Q, respectively with characteristics,

$$\begin{aligned} T: &\quad \Lambda^1 \times \Lambda^1 \longmapsto \Lambda^1 \\ Q: &\quad \Lambda^1 \times \Lambda^1 \times \Lambda^1 \longmapsto \mathbb{R} \end{aligned}$$

by the following equations

$$\begin{aligned} T(X,Y) &:= \nabla_X Y - \nabla_Y X - [X,Y] \\ Q(X,Y,Z) &:= (\nabla_X g)(Y,Z) \end{aligned}$$

where $X,Y,Z \in \Lambda^1$ are vectorfields on $\mathscr{M}$.

A general connection on a manifold with non-degenerate metric can be decomposed into the Levi-Civita connection and an arbitrary $(2,1)$-tensor. The dimension of this tensor is therefore m^3 where m is the dimension of the manifold $\mathscr{M}$. Below it is shown that it can be decomposed into one part containing only the torsion tensor, T, and one part containing only the non-metricity tensor, Q. These two tensors have the dimensions $\frac{1}{2}m^2(m-1)$ and $\frac{1}{2}m^2(m+1)$ respectively which together give m^3. The torsion do not appear directly in the connection but as the contorsion and that is also the case with the non-metricity tensor. The following notation will be used in what follows,

$$^\flat T(X,Y,Z) := g(T(X,Y),Z)$$

In the next proposition the contorsion and con-metricity tensors are defined.

Definition 2.2
Let ∇ be a connection in a manifold $\mathscr{M}$ with non-degenerate metric g. Define the contorsion tensor, S, and the con-metricity tensor, P, respectively, with same characteristics,

$$S, P: \quad \Lambda^1 \times \Lambda^1 \times \Lambda^1 \longmapsto \mathbb{R}$$

by following equations

$$\begin{aligned} ^\flat S(X,Y,Z) &:= \frac{1}{2}(^\flat T(X,Y,Z) - {}^\flat T(Y,Z,X) + {}^\flat T(Z,X,Y)) \\ ^\flat P(X,Y,Z) &:= \frac{1}{2}(-Q(X,Y,Z) - Q(Y,Z,X) + Q(Z,X,Y)) \end{aligned}$$

where $X,Y,Z \in \Lambda^1$ are vectorfields on $\mathscr{M}$ and $S(X,Y) = g^{-1}(^\flat S(X,Y,\cdot))$, $P(X,Y) = g^{-1}(^\flat P(X,Y,\cdot))$.

Now, any connection can be expressed in terms of the Levi Civita connection with respect to a non-degenerate metric, denoted by ${}^g\nabla$, plus the contorsion and the con-metricity tensors defined above, i.e,

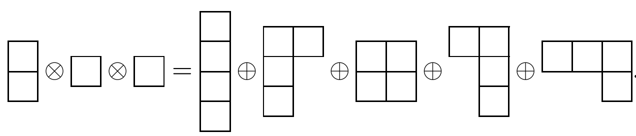
Proposition 2.3
Let ∇ be an arbitrary connection on a manifold $\mathscr{M}$, let further g be a non-degenerate metric on $\mathscr{M}$ and ${}^g\nabla$ corresponding Levi-Civita connection. Let S, P be the tensors defined in 2.2 .Then

$$\nabla_X Y = {}^g\nabla_X Y + S(X,Y) + P(X,Y)$$

The curvature tensor of an arbitrary connection, is defined by,

$$R(X,Y)Z := [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \tag{1}$$

will no longer take values in the lie algebra $\mathfrak{o}(m)$ as does the curvature tensor of the Levi-Civita connection, but (will in the generic case take values) in $\mathfrak{gl}(m)$. The identities of the curvature tensor will therefor be altered, and its irreducible parts look in the generic case like



(2)

Proposition 2.4
The four identities of the Riemann curvature tensor of an arbitrary connection are

$$\begin{aligned}
&(i) && R_{(ab)c}{}^{d} = 0\\
&(ii) && R_{[abc]}{}^{d} = \nabla_{[a}T_{bc]}{}^{d} - T_{[ab}{}^{e}T_{c]e}{}^{d}\\
&(iii) && R_{ab(cd)} = -(\nabla_{[a}Q)_{b]cd} - \frac{1}{2}T_{ab}{}^{e}Q_{ecd}\\
&(iv) && R_{abcd} - R_{cdab} = \frac{3}{2}(R_{[abc]d} + R_{[bcd]a} - R_{[cda]b} - R_{[dab]c}) +\\
& && \quad + R_{ab(cd)} - R_{bc(da)} - R_{cd(ab)} + R_{da(bc)} + R_{ac(db)} - R_{db(ac)}
\end{aligned}$$

By the skew-tableaux (the two tableaux on the right in equation (2) above) it is stressed that these two irreducible parts will vanish when the connection is metric, i.e, the right hand side of identity 3 vanishes. In the generic case there are two possible contractions that can be made.

Definition 2.5
Let ∇ be an arbitrary connection on a manifold $\mathscr{M}$ with curvature tensor, R. From the curvature it is possible to construct two types of $(2,0)$ tensors by contraction, namely,

$$\begin{aligned}
R_{ab} &:= R_{acb}{}^{c},\\
V_{ab} &:= R_{abc}{}^{c}.
\end{aligned}$$

The first one is the generalized Ricci tensor and the second one will here be refered to as the Schouten two-form.

The identities of the Ricci and Schouten tensors can be read off directly from the original curvature identities.

Proposition 2.6
Let R_{ab} be the Ricci tensor and V_{ab} the Schouten tensor of an arbitrary connection, ∇, then the second and third curvature identities implythe relations,

$$\begin{aligned} 2R_{[ab]} &= V_{ab} - \nabla_c T_{ab}{}^c - 2\nabla_{[a}T_{b]} - T_{ab}{}^c T_c, \\ V_{ab} &= -(\nabla_{[a}Q)_{b]} - \frac{1}{2}T_{ab}{}^c Q_c \end{aligned}$$

The only integrability conditions to the curvature identities are the Bianchi identities.

Proposition 2.7
Let ∇ be an arbitrary connection with curvature tensor R, and torsion tensor T. Then the Bianchi identity reads

$$\nabla_{[a}R_{bc]d}{}^e = T_{[ab}{}^f R_{c]fd}{}^e$$

from which the identities involving the Ricci tensor R_{ab}, and the Schouten two-form V, are derived,

$$\begin{aligned} 2\nabla_{[a}R_{b]c} + T_{ab}{}^d R_{dc} &= \nabla_d R_{abc}{}^d - 2T_{d[a}{}^e R_{b]ec}{}^d \\ dV &= 0 \end{aligned}$$

2.2 Conformal transformations

Below, conformal transformations in the case of an arbitrary connection are studied. There will be some changes compared to the ordinary Levi Civita case when the connection involves torsion and non-metricity.

Definition 2.8
Let $\mathscr{M}$ be a manifold with metric g, let further ∇ be an arbitrary connection on $\mathscr{M}$. Let ${}^c g := e^{2\phi}g$ denote a conformal transformation then define the **conformal tensor**, *denoted by $\mathscr{C}$, with characteristics*

$$\mathscr{C}: \qquad \Lambda^1 \times \Lambda^1 \longmapsto \Lambda^1,$$

by

$$\mathscr{C}(X,Y) := {}^c\nabla_X Y - \nabla_X Y$$

where $X, Y \in \Lambda^1$ are vectorfields on $\mathscr{M}$.

By a straight forward calculation one ends up with the transformations of the characteristic tensors of a connection under conformal transformation.

Proposition 2.9
Let $\mathscr{M}$ be a manifold with metric g and ∇ be an arbitrary connection on $\mathscr{M}$. Let R, T, Q denote the Riemann, torsion and non-metricity tensors respectively. Then their transformations under a conformal transformation can be expressed in terms of the conformal tensor, $\mathscr{C}$, as

$$\begin{aligned}
{}^{c}\!R(X,Y)Z - R(X,Y)Z =& (\nabla_X \mathscr{C})(Y,Z) - (\nabla_Y \mathscr{C})(X,Z) + \\
& + \mathscr{C}(X, \mathscr{C}(Y,Z)) - \mathscr{C}(Y, \mathscr{C}(X,Z)) + \mathscr{C}(T(X,Y),Z), \\
{}^{c}T(X,Y) - T(X,Y) =& \mathscr{C}(X,Y) - \mathscr{C}(Y,X), \\
{}^{c}Q(X,Y,Z) - e^{2\phi}Q(X,Y,Z) =& e^{2\phi}[2X[\phi]g(Y,Z) - g(\mathscr{C}(X,Y),Z) - g(Y,\mathscr{C}(X,Z))]
\end{aligned}$$

In the case of the Levi-Civita connection the conformal tensor is most easily extracted from the above proposition.

Proposition 2.10
Let ${}^{g}\nabla$ be the Levi-Civita connection on a manifold with non-degenerate metric g, then its conformal tensor, denoted by ${}^{g}\mathscr{C}$, reads

$${}^{g}\mathscr{C}(X,Y) = X[\phi]Y + Y[\phi]X - g(X,Y)^{\sharp}d\phi$$

From these two propositions the conformal tensor in the generic case can be derived.

Proposition 2.11
Let ∇ be an arbitrary connection on a manifold $\mathscr{M}$ with non-degenerate metric g, let further S, P denote the contorsion and the con-metricity tensor respectively. Then the conformal tensor of ∇ reads

$$\mathscr{C}(X,Y) = {}^{g}\mathscr{C}(X,Y) + {}^{c}S(X,Y) - S(X,Y) + {}^{c}P(X,Y) - P(X,Y)$$

3 The connections associated with an almost product structure

Here a quick review on the concepts of almost product structures will be given, for a more thorough treatise see refs. [1, 6].

Notation 3.1
We will denote the objects on our space with an underline, i.e.,

$\underline{\mathscr{M}}$	*Manifold*
$T\underline{\mathscr{M}}$	*Tangent bundle of* $\underline{\mathscr{M}}$
$T^*\underline{\mathscr{M}}$	*Cotangent bundle of* $\underline{\mathscr{M}}$
$\underline{g}$	*Metric on* $\underline{\mathscr{M}}$
$\underline{d}$	*Exterior derivative*
$\underline{X}$	*Vector field on* $\underline{\mathscr{M}}$

to list the primarily used objects. We will use this underlining principle for all objects on $\underline{\mathscr{M}}$ *whenever there may be risk of confusion.*

Definition 3.2
Let I *be an almost product structure on a manifold* $\underline{\mathscr{M}}$ *with riemannian metric* $\underline{g}$ *and let* $X,\ Y \in T\underline{\mathscr{M}}$ *be vector fields. Then the triplet* $(\underline{\mathscr{M}},\ \underline{g},\ I)$ *is called an* **riemannian almost product structure** *or simply an* **almost product manifold** *if*

$$\underline{g}(IX, IY) = \underline{g}(X, Y)$$

or in other words, I *is a automorphism of* $\underline{g}$ *in the sense that the following diagram commutes:*

$$\begin{array}{ccc} T\underline{\mathscr{M}} & \xrightarrow{I} & T\underline{\mathscr{M}} \\ \downarrow{\scriptstyle \underline{g}} & & \downarrow{\scriptstyle \underline{g}} \\ T^*\underline{\mathscr{M}} & \xrightarrow[I^t]{} & T^*\underline{\mathscr{M}} \end{array}$$

i.e.,

$$I^t \circ \underline{g} \circ I = \underline{g}$$

Proposition 3.3
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ *define a riemannian almost product structure on* $\underline{\mathscr{M}}$ *with* $\dim \underline{\mathscr{M}} = \underline{m}$*, then*

(i). $I^2 = 1$

(ii). All eigenvalues are ± 1.

(iii). $\mathrm{tr} I = 2k - \underline{m}$*, where* k *is the number of positive eigenvalues.*

(iv). $I \in Gr(k, \underline{m}) \equiv O(\underline{m})/(O(k) \times O(\underline{m} - k))$.

(v). There is a preferred base called the oriented base in which I *is diagonal and ordered, i.e., it takes the form*

$$I = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$$

The almost product structure will serve as a rigging of the tangentbundle by looking at the spaces of eigenvectors to the almost product structure.

Definition 3.4
Let I be an almost product structure on $\underline{\mathscr{M}}$, then I defines two natural distributions of $T\underline{\mathscr{M}}$, denoted $\mathcal{D}$ and $\mathcal{D}'$ respectively, in the following way. Let

$$\begin{aligned} \mathcal{D}_x &:= \{X \in T_x\underline{\mathscr{M}} : IX = X\}, \\ \mathcal{D}'_x &:= \{X \in T_x\underline{\mathscr{M}} : IX = -X\}, \end{aligned}$$

then

$$\mathcal{D} := \bigcup_{x\in\underline{\mathscr{M}}} \mathcal{D}_x, \quad \mathcal{D}' := \bigcup_{x\in\underline{\mathscr{M}}} \mathcal{D}'_x.$$

Seen as an endomorphism of the tangent bundle two projection operators can be formed from the almost product structure as it squares to one. These will now be projective mappings from the tangent bundle to these two sub-bundles defined above.

Definition 3.5
From an almost product structure I on a manifold $\underline{\mathscr{M}}$ we can define two projection operators through

$$\begin{aligned} \mathcal{P} &:= \frac{1}{2}(1+I) \\ \mathcal{P}' &:= \frac{1}{2}(1-I). \end{aligned}$$

These will be mappings in the sense $\mathcal{P} : T\underline{\mathscr{M}} \to \mathcal{D}$ and $\mathcal{P}' : T\underline{\mathscr{M}} \to \mathcal{D}'$ respectively.

The Riemann metric in the triplet of a almost product manifold will now split into two parts which will be the induced metrics on these two sub-bundles of the tangent bundle.

Definition 3.6
Let $\underline{\mathscr{M}}$ be a riemannian or pseudo-riemannian manifold with metric, $\underline{g}$, I a reflective structure with $\mathcal{P}$ and $\mathcal{P}'$ the corresponding projectors, then define the two associated metrics with respect to the reflective structure by

$$g(X,Y) := \underline{g}(\mathcal{P}X, \mathcal{P}Y), \quad g'(X,Y) := \underline{g}(\mathcal{P}'X, \mathcal{P}'Y)$$

which implies that $\underline{g}$ splits into these two parts, i.e.,

$$\underline{g} = g + g'.$$

3.1 Tensors associated with an almost product structure

There is one main tensor in the context of almost product manifolds and that is the deformation tensor. This tensor is most suitably decomposed into two irreducible parts namely the Nijenhuis tensor and the Jordan tensor.

Definition 3.7
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define a riemannian almost product structure and define the **Nijenhuis tensor** *as the measure of how much $\underline{d}_I$ fails to be a coboundary operator. The Nijenhuis tensor is thus a (2,1) tensor. Let $X, Y \in \Lambda^1$ be vector fields on $\underline{\mathscr{M}}$, then the characteristics of the Nijenhuis tensor are*

$$N_I(X, Y) : \Lambda^1 \times \Lambda^1 \longmapsto \Lambda^1$$

and we define it through the quadratic action of d_I on functions $f \in C^\infty(\underline{\mathscr{M}})$,

$$< -N_I(X, Y), df >:= \underline{d}_I \underline{d}_I f(X, Y).$$

It follows that the Nijenhuis tensor measures the failure in closure of the operator d_I and can thus be considered as a kind of torsion. Alternatively, as the equivalent definition below shows, it measures the curvature of the endomorphism, i.e.,

$$N_I[X, Y] := I([X, Y]_I) - [I(X), I(Y)]$$

Alternatively the Nijenhuis tensor can be seen as measuring how far this endomorphism is from being a Lie algebra homomorphism of the infinite-dimensional Lie algebra of vector fields on $\mathscr{M}$.

Definition 3.8
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define a riemannian almost product structure, and let $\{\cdot, \cdot\}$ be the Jordan bracket. The **Jordan tensor** *associated to I, denoted M_I, with the following characteristics:*

$$M_I : \Lambda^1 \times \Lambda^1 \longmapsto \Lambda^1$$

is defined by,

$$M_I(X, Y) := I\{X, Y\}_I - \{IX, IY\}$$

where $X, Y \in \Lambda^1$ are vector fields on $\underline{\mathscr{M}}$. The analogy to the Nijenhuis tensor is obvious .

Both the Nijenhuis and the Jordan tenor can be expressed entirely in terms of the covariant derivative of the almost product structure.

Definition 3.9
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define a riemannian almost product structure. Let $\underline{\nabla}$ be the Levi–Civita connection on $\underline{\mathscr{M}}$ and define the **deformation tensor** *associated with the endomorphism I, denoted H_I, with the following characteristics:*

$$H_I : \Lambda^1 \times \Lambda^1 \longmapsto \Lambda^1$$

H_I is defined by the expression

$$H_I(X, Y) := (I\underline{\nabla}_X I - \underline{\nabla}_{IX} I)(Y),$$

where $X, Y \in \Lambda^1$ are two vector fields on $\underline{\mathscr{M}}$. An equivalent definition is given by,

$$H_I(X, Y) := N_I(X, Y) + M_I(X, Y).$$

Looking at the characteristic tensors of a distribution the deformation tensor of an almost product structure can now be decomposed into the deformation tensors of the two complementary distributions defined by the almost product structure.

Definition 3.10
Let $\mathcal{D}$ be a k-distribution with projection $\mathcal{P}$ on a riemannian manifold $\underline{\mathscr{M}}$ with non-degenerate metric $\underline{g}$. Let $\underline{\nabla}$ be the Levi–Civita connection with respect to this metric and let $\mathcal{P}' := 1 - \mathcal{P}$ be the coprojection of $\mathcal{D}$. Now define the following tensors with characteristics

$$\begin{aligned} H, L, K : &\quad \Lambda^1_{\mathcal{D}} \times \Lambda^1_{\mathcal{D}} \longmapsto \Lambda^1_{\mathcal{D}'} \\ \kappa : &\quad \Lambda^1_{\mathcal{D}'} \longmapsto \mathbb{R} \end{aligned}$$

and

$$\begin{aligned} (i) &\quad H(X,Y) := \mathcal{P}'\underline{\nabla}_{\mathcal{P}X}\mathcal{P}Y && \textbf{deformation tensor}, \\ (ii) &\quad L(X,Y) := \frac{1}{2}(H(X,Y) - H(Y,X)) && \textbf{twisting tensor}, \\ (iii) &\quad K(X,Y) := \frac{1}{2}(H(X,Y) + H(Y,X)) && \textbf{extrinsic curvature tensor}, \\ (iv) &\quad {}^\sharp\kappa := \mathrm{tr} H && \textbf{mean curvature tensor}, \\ (v) &\quad W(X,Y) := K(X,Y) - \frac{1}{k}{}^\sharp\kappa g(X,Y) && \textbf{conformation tensor}. \end{aligned}$$

This gives us the decomposition of the deformation tensor in its anti-symmetric, symmetric-traceless and trace parts accordingly,

$$H = L + W + \frac{1}{k}{}^\sharp\kappa g.$$

The extrinsic curvature tensor and the twisting tensor can be written in a more elegant fashion.

Proposition 3.11
Let $\mathcal{D}$ be a distribution on a manifold $\underline{\mathscr{M}}$ with metric $\underline{g}$, let further $g(X,Y) = \underline{g}(\mathcal{P}X, \mathcal{P}Y)$ be the induced metric on the distribution, then the symmetric part of the deformation tensor can be written like

$$K(X,Y)(\varphi) = -\frac{1}{2}\mathscr{L}_{\sharp\varphi'} g(X,Y), \ \ or \ \ {}^\flat K(X,Y,Z) = -\frac{1}{2}\mathscr{L}_{Z'} g(X,Y),$$

where the prime denotes projection along the normal directions by $\mathcal{P}'$. The relation for the anti-symmetric part on the other hand is

$$L(X,Y) = \frac{1}{2}\mathcal{P}'[\mathcal{P}X, \mathcal{P}Y]$$

The conformal properties of the irreducible parts of the deformation tensor can be found in next proposition.

Proposition 3.12
Let $\underline{\mathscr{M}}$ be a riemannian manifold with metric $\underline{g}$, let I be an almost product structure on $\underline{\mathscr{M}}$ which split the metric in $\underline{g} = g + g'$ and let $\lambda = e^{2\phi}$ be a conformal transformation on g, i.e., ${}^c\underline{g} = \lambda\underline{g}$ then the symmetric parts of the deformation tensor will transform like

$$\begin{aligned} {}^cK(\varphi) &= K(\varphi) - \frac{1}{2}\lambda^{-1\sharp}\varphi'[\lambda]g = K(\varphi) - {}^\sharp\varphi[\phi]g \\ {}^c\kappa(X) &= \kappa(X) - \frac{1}{2}k\lambda^{-1}X'[\lambda] = \kappa(X) - kX'[\phi] \\ {}^cW &= W \\ {}^cL &= L \end{aligned}$$

Now denoting the deformation tensor of the complementary distribution $\mathcal{D}'$ by H' and its irreducible parts by L', K', κ', W' respectively we can express the Nijenhuis tensor and the Jordan tensor in terms of these characteristic tensors.

Lemma 3.13
Let I be an almost product structure on a manifold $\underline{\mathscr{M}}$ and let its associated projection operators be $\mathcal{P} := \frac{1}{2}(1+I)$, $\mathcal{P}' := \frac{1}{2}(1-I)$, then

$$\begin{aligned} &(i) & N_{\mathcal{P}} &= N_{\mathcal{P}'} \\ &(ii) & N_I &= 4N_{\mathcal{P}} \\ &(iii) & \frac{1}{2}[\mathcal{P}, \mathcal{P}'] &= N_{\mathcal{P}} \\ &(iv) & N_{\mathcal{P}}(X,Y) &= -\mathcal{P}'[\mathcal{P}X, \mathcal{P}Y] - \mathcal{P}[\mathcal{P}'X, \mathcal{P}'Y] \end{aligned}$$

Proposition 3.14
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define an riemannian almost product structure and let L, L' be the twisting tensors of the distributions defined by I. Then

$$\frac{1}{8}N_I = -L - L'$$

Lemma 3.15
Let I be an almost product structure on a manifold $\underline{\mathscr{M}}$ and let its associated projection operators be $\mathcal{P} := \frac{1}{2}(1+I)$, $\mathcal{P}' := \frac{1}{2}(1-I)$. Let M denote the Jordan tensor, then

$$\begin{aligned} &(i) & M_{\mathcal{P}} &= M_{\mathcal{P}'} \\ &(ii) & M_I &= 4M_{\mathcal{P}} \\ &(iii) & \frac{1}{2}\{\mathcal{P}, \mathcal{P}'\} &= M_{\mathcal{P}} \\ &(iv) & M_{\mathcal{P}}(X,Y) &= -\mathcal{P}'\{\mathcal{P}X, \mathcal{P}Y\} - \mathcal{P}\{\mathcal{P}'X, \mathcal{P}'Y\} \end{aligned}$$

Proposition 3.16
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define a riemannian almost product structure, K, K' be the extrinsic curvature tensors of the distributions defined by I, then

$$\frac{1}{8} M_I = -K - K'$$

3.2 Three relevant connections

In a almost product manifold there are three different connections of importance, which will be defined in this subsection. The first is of course the Levi–Civita connection, from which the other two will be defined by simply adding a tensor to it. These will be refered to as the adapted and the Vidal connection. Their basic feature is that they commute with the almost product structure which means that they respect the rigging of the tangent space defined by the almost product structure. The additional feature of the adapted connection is that it is metric which together with the above feature implies that it respects the induced metrics on the two characteristic distributions associated with the almost product structure. The Vidal connection which is metric iff the characteristic distributions are geodesic will play an important role when looking at gauge theories and fiber bundles since they need no metric in the total space and are of the type (GF, GD). By adding the group metric to the fiber we can construct an almost product manifold in which the Vidal connection will reduce to the gauge covariant derivative. This will be explicitly done in section 5. The curvature components of the Vidal connections lying entirely in the tensor algebra of the characteristic distributions, also called the semi-basic parts, will in a more natural way measure the curvature in the respective distributions. This is due to the fact that it does not depend on the connections in its co-parts. What this means explicitly will become clear when the relations are derived.

Definition 3.17
Let $\underline{\mathscr{M}}$ be a riemannian or pseudo-riemannian manifold with non-degenerate metric $\underline{g}$ and corresponding Levi–Civita connection $\underline{\nabla}$. Let I define distributions as in definition 3.4. Then the following two definitions of the **adapted connection** *are equivalent*

(i). $\tilde{\underline{\nabla}}_X Y := \underline{\nabla}_X Y + A(X,Y), \quad A(X,Y) := \frac{1}{2} I \underline{\nabla}_X I(Y)$

(ii). $\tilde{\underline{\nabla}}_X Y := \mathcal{P}\underline{\nabla}_X \mathcal{P}Y + \mathcal{P}'\underline{\nabla}_X \mathcal{P}'Y$

Definition 3.18
Let $\underline{\mathscr{M}}$ be a riemannian or pseudo-riemannian manifold with non-degenerate metric $\underline{g}$ and corresponding Levi–Civita connection $\underline{\nabla}$, let I define a foliation. Then the **Vidal connection** *is defined by*

$$\tilde{\tilde{\underline{\nabla}}}_X Y := \tilde{\underline{\nabla}}_X Y + B(X,Y), \quad B(X,Y) := \frac{1}{4}(\underline{\nabla}_{IY} I + I \underline{\nabla}_Y I)(X).$$

Proposition 3.19
Let $\underline{X}, \underline{Y}$ be vectorfields on an almost product manifold $(\underline{\mathscr{M}}, \underline{g}, I)$, let $X = \mathcal{P}\underline{X}$, $X' = \mathcal{P}'\underline{X}$ and similar for $\underline{Y}$. Then the Vidal connection can be written

$$\underline{\tilde{\tilde{\nabla}}}_{\underline{X}}\underline{Y} = \begin{pmatrix} \mathcal{P}\underline{\nabla}_X Y & \mathcal{P}'[X, Y'] \\ \mathcal{P}[X', Y] & \mathcal{P}'\underline{\nabla}_{X'} Y' \end{pmatrix}$$

The two recently introduced tensors, A and B, are in fact related.

Proposition 3.20
Let B be the tensor defined in 3.18 and A the tensor defined in 3.17, then it is possible express the tensor B in terms of A and the almost product structure I as

$$B(X, Y) = \frac{1}{2}\left(A(Y, X) - A(IY, IX)\right).$$

The most important property of the two connections defined above, is that they both commute with the almost product structure.

Proposition 3.21
Let $\underline{\tilde{\nabla}}$ denote the adapted connection defined in 3.17 and $\underline{\tilde{\tilde{\nabla}}}$ the Vidal connection defined in 3.18, then their principal feature is that they both commute with the almost product structure I, i.e.,

$$\underline{\tilde{\nabla}}_X I = \underline{\tilde{\tilde{\nabla}}}_X I = 0$$

Only one of them though will, in the generic case, be metric and that is the adapted connection.

Proposition 3.22
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ be a riemannian almost product structure on $\underline{\mathscr{M}}$ and $\underline{\tilde{\nabla}}$ the adapted connection defined in 3.17, then this connection is metric with respect to the splitting of $\underline{g}$ according to 3.6, i.e.,

$$\underline{\tilde{\nabla}} g = 0$$
$$\underline{\tilde{\nabla}} g' = 0$$

The connection components takes a most pleasant form in the oriented basis. The notation,

$$\underline{E}_{\bar{a}} = (\underline{E}_a, \underline{E}_{a'})$$

will be used, where unprimed(primed) index denotes the basis of the characteristic unprimed(primed) distribution.

Proposition 3.23
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define a riemannian almost product structure, let $\underline{\omega}$, $\underline{\tilde{\omega}}$ and $\underline{\tilde{\tilde{\omega}}}$ denote the connection one-forms of the Levi–Civita connection, the adapted connection and the Vidal connection respectively. Let furthermore H, H' denote the deformation tensors with respect to I and C, C' be coefficients of anholonomy Then

$$\begin{aligned}
\underline{\omega} &= \left[\begin{pmatrix} \omega & H \\ -H^t & \Omega \end{pmatrix}, \begin{pmatrix} \Omega' & H' \\ -H'^t & \omega' \end{pmatrix}\right] \\
\underline{\tilde{\omega}} &= \left[\begin{pmatrix} \omega & 0 \\ 0 & \Omega \end{pmatrix}, \begin{pmatrix} \Omega' & 0 \\ 0 & \omega' \end{pmatrix}\right] \\
\underline{\tilde{\tilde{\omega}}} &= \left[\begin{pmatrix} \omega & 0 \\ 0 & C \end{pmatrix}, \begin{pmatrix} C' & 0 \\ 0 & \omega' \end{pmatrix}\right]
\end{aligned}$$

The coefficients of anholonomy are defined by $[E_{\bar{a}}, E_{\bar{b}}] =: C_{\bar{a}\bar{b}}{}^{\bar{c}} E_{\bar{c}}$ and can be used to express all the connection components and the deformation tensors. It is interesting to note their behaviour under a local $O(k) \times O(k')$ transformation:

$$\begin{aligned}
\tilde{C}_{ab}{}^{c} &= u_a{}^d u_b{}^e C_{de}{}^f u^{-1}_f{}^c + 2\tilde{E}_{[a}[u_{b]}{}^f]u^{-1}_f{}^c, && \text{Coefficients of anholonomy} \\
\tilde{C}_{ab}{}^{c'} &= u_a{}^d u_b{}^e C_{de}{}^{f'} u^{-1}_{f'}{}^{c'}, && \text{Tensor} \\
\tilde{C}_{a'b}{}^{c} &= u_{a'}{}^{d'} u_b{}^e C_{d'e}{}^f u^{-1}_f{}^c + \tilde{E}_{a'}[u_b{}^f]u^{-1}_f{}^c, && \text{Connection}
\end{aligned}$$

This is what enables the defininition of the Vidal connection, i.e. the observation that the $C_{ab'}{}^{c'}, C_{a'b}{}^{c}$ parts of the structure coefficients transform as connections under local $O(k) \times O(k')$ transformations.

Proposition 3.24
Let ω, Ω, H, L, K be the components of the Levi-Civita connection and C the coefficients of anholonomy, then they are related through the following relations,

$$\begin{aligned}
C_{abc} &= 2\omega_{[ab]c} \\
C_{a'bc} &= \Omega'_{a'bc} + H_{bca'} \\
C_{abc'} &= 2L_{abc'} \\
C_{a'b'c} &= 2L'_{a'b'c} \\
C_{ab'c'} &= \Omega_{ab'c'} + H'_{b'c'a} \\
C_{a'b'c'} &= 2\omega'_{[a'b']c'}
\end{aligned}$$

with inverse relations

$$
\begin{aligned}
\omega_{abc} &= C_{a[bc]} - \frac{1}{2}C_{bca} = \frac{1}{2}C_{abc} + C_{c(ab)} \\
\Omega'_{a'bc} &= C_{a'[bc]} - \frac{1}{2}C_{bca'} \\
K_{abc'} &= C_{c'(ab)} \\
L_{abc'} &= \frac{1}{2}C_{abc'} \\
H_{abc'} &= \frac{1}{2}C_{abc'} + C_{c'(ab)} \\
H'_{a'b'c} &= \frac{1}{2}C_{a'b'c} + C_{c(a'b')} \\
L'_{a'b'c} &= \frac{1}{2}C_{a'b'c} \\
K'_{a'b'c} &= C_{c(a'b')} \\
\Omega_{ab'c'} &= C_{a[b'c']} - \frac{1}{2}C_{b'c'a} \\
\omega'_{a'b'c'} &= C_{a'[b'c']} - \frac{1}{2}C_{b'c'a'} = \frac{1}{2}C_{a'b'c'} + C_{c'(a'b')}
\end{aligned}
$$

4 The curvature components and their relations

As fiber bundles and fibrations are examples of almost product manifolds with the additional property of the existense of a surjective submersion of the total space down to a base space, it would be interesting to see what parts of the Vidal and adapted connections, which are defined on the total space, that survive under this submersion. In the total space two new differential operators were defined in [1].

Definition 4.1
Let I be an almost product structure on a manifold $\underline{\mathscr{M}}$ with exterior derivative $\underline{d}$. Let furthermore $\underline{d}_I$ denote the exterior derivative associated with I and define two new differential operators by

$$
\begin{aligned}
d &:= \frac{1}{2}(\underline{d} + \underline{d}_I) \\
d' &:= \frac{1}{2}(\underline{d} - \underline{d}_I)
\end{aligned}
$$

An equivalent definition is by the two projection operators defined by the endomorphism I, $\mathcal{P} := \frac{1}{2}(1+I)$ and $\mathcal{P}' := \frac{1}{2}(1-I)$, then the operators are simply $d \equiv \underline{d}_{\mathcal{P}}$ and $d' \equiv \underline{d}_{\mathcal{P}'}$.

These differential operators become coboundary operators if and only if the Nijenhuis tensor vanishes, which is the same as to say that both the characteristic distributions of the almost product structure are integrable. In a fibration for instance this is not normally true, except for the trivial case of a product manifold, so we will keep track of all components surviving the submersion and those who will not. When it comes to these differential operators it is therefor clear that d defined above will in general differ from the exterior derivative defined on the base space. By projecting out the semi-basic parts of all quantities we can keep track of the parts that survives the submersion.

Definition 4.2
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ denote a riemannian almost product structure and $\mathcal{D}$, $\mathcal{D}'$ be the associated distributions then define the brackets associated with these distributions with following characteristics

$$\begin{aligned}[\,\cdot\,,\,\cdot\,]^{\mathcal{D}}: &\quad \Lambda^1_{\mathcal{D}} \times \Lambda^1_{\mathcal{D}} \longmapsto \Lambda^1_{\mathcal{D}},\\ [\,\cdot\,,\,\cdot\,]^{\mathcal{D}'}: &\quad \Lambda^1_{\mathcal{D}'} \times \Lambda^1_{\mathcal{D}'} \longmapsto \Lambda^1_{\mathcal{D}'},\end{aligned}$$

by

$$\begin{aligned}[X,Y]^{\mathcal{D}} &:= \mathcal{P}[\mathcal{P}X, \mathcal{P}Y],\\ [X,Y]^{\mathcal{D}'} &:= \mathcal{P}'[\mathcal{P}'X, \mathcal{P}'Y],\end{aligned}$$

where $X, Y \in \Lambda^1$ are vectorfields on $\underline{\mathscr{M}}$.

Here it is clear that the twisting tensor that measures the amount of non-commutativity is non-semi-basic. The two brackets defined above will therefor not satisfy the Jacobi identity in the total space, but will differ with some terms involving the twisting tensors. The same procedure can be made to define the semi-basic torsions and curvature tensor of a distribution.

Definition 4.3
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ denote a riemannian almost product structure and $\mathcal{D}$, $\mathcal{D}'$ be the associated distributions. Let further $T^p_q(T\underline{\mathscr{M}})$ denote the set of (p,q)-tensors on $\underline{\mathscr{M}}$ and $T^p_q(\mathcal{D})$ $(T^p_q(\mathcal{D}'))$ denote the set of tensors lying entirely in $\mathcal{D}$ $(\mathcal{D}')$. Now define the associated Levi–Civita connections with following characteristics

$$\begin{aligned}\underline{\nabla}^{\mathcal{D}}_X: &\quad T^p_q(T\underline{\mathscr{M}}) \longmapsto T^p_q(\mathcal{D}),\\ \underline{\nabla}^{\mathcal{D}'}_X: &\quad T^p_q(T\underline{\mathscr{M}}) \longmapsto T^p_q(\mathcal{D}'),\end{aligned}$$

through

$$\begin{aligned}\underline{\nabla}^{\mathcal{D}}_X Y &:= \mathcal{P}\underline{\nabla}_{\mathcal{P}X}\mathcal{P}Y,\\ \underline{\nabla}^{\mathcal{D}'}_X &:= \mathcal{P}'\underline{\nabla}_{\mathcal{P}'X}\mathcal{P}'Y,\end{aligned}$$

where $X, Y \in \Lambda^1$ are vectorfields on $\underline{\mathscr{M}}$. Further define the torsion and curvature of the corresponding connections by

$$\begin{aligned}T^{\mathcal{D}}(X,Y) &:= \underline{\nabla}^{\mathcal{D}}_X Y - \underline{\nabla}^{\mathcal{D}}_Y X - [X,Y]^{\mathcal{D}},\\ T^{\mathcal{D}'}(X,Y) &:= \underline{\nabla}^{\mathcal{D}'}_X Y - \underline{\nabla}^{\mathcal{D}'}_Y X - [X,Y]^{\mathcal{D}'},\end{aligned}$$

and

$$\begin{aligned}R^{\mathcal{D}}(X,Y)Z &:= \underline{\nabla}^{\mathcal{D}}_X \underline{\nabla}^{\mathcal{D}}_Y Z - \underline{\nabla}^{\mathcal{D}}_Y \underline{\nabla}^{\mathcal{D}}_X Z - \underline{\nabla}^{\mathcal{D}}_{[X,Y]^{\mathcal{D}}} Z,\\ R^{\mathcal{D}'}(X,Y)Z &:= \underline{\nabla}^{\mathcal{D}'}_X \underline{\nabla}^{\mathcal{D}'}_Y Z - \underline{\nabla}^{\mathcal{D}'}_Y \underline{\nabla}^{\mathcal{D}'}_X Z - \underline{\nabla}^{\mathcal{D}'}_{[X,Y]^{\mathcal{D}'}} Z.\end{aligned}$$

In this case it is clear that the curvature defined above, will not in general be tensorial in the latter indices. As before these non-tensorial parts will vanish under the submersion. The torsion will still be tensorial though.

4.1 The Vidal connection

In this subsection, all tensors associated with the Vidal connection will be derived, that is the torsion tensor, the non-metricity tensor, the Riemann tensor and its traces. The curvature identities are used to express all components but the two totally semi-basic ones only in terms of the different irreducible parts of the deformation tensor and its derivatives. From these curvature identities there also arise a couple of new relations involving only parts of the deformation tensor. Two of these will evidently become the Bianchi identity of the two twisting tensors but two others will appear in a more unfamiliar fashion.

From the definition it is clear that the Vidal connection is neither torsion-free, nor metric in the generic case of an almost product manifold. It is therefor interesting to see what the torsion and non-metricity tensor look like in this case. In ref. [1] the following proposition was derived.

Proposition 4.4
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define an almost product manifold, let N_I denote the Nijenhuis tensor of I and $\underline{\tilde{\tilde{\nabla}}}$ denote the Vidal connection defined in 3.18, then,

$$\frac{1}{4}N_I(X,Y) = \underline{\tilde{\tilde{T}}}(X,Y).$$

Together with proposition 3.14 the torsion tensor can be written in component form.

Proposition 4.5
Let $\underline{\tilde{\tilde{T}}}$ be the torsion tensor of the Vidal connection, $\underline{\tilde{\tilde{\nabla}}}$, then in component form it reads

$$\begin{aligned}
\underline{\tilde{\tilde{T}}}_{ab}{}^{c} &= 0,\\
\underline{\tilde{\tilde{T}}}_{ab}{}^{c'} &= -2L_{ab}{}^{c'},\\
\underline{\tilde{\tilde{T}}}_{a'b}{}^{c} &= 0,\\
\underline{\tilde{\tilde{T}}}_{ab'}{}^{c'} &= 0,\\
\underline{\tilde{\tilde{T}}}_{a'b'}{}^{c} &= -2L'_{a'b'}{}^{c},\\
\underline{\tilde{\tilde{T}}}_{a'b'}{}^{c'} &= 0.
\end{aligned}$$

and

$$\begin{aligned}
\underline{\tilde{\tilde{T}}}_{a} &= 0,\\
\underline{\tilde{\tilde{T}}}_{a'} &= 0.
\end{aligned}$$

As was seen in [1] the torsion tensor measures the non-integrability of the two complementary distributions defined by an almost product structure. The non-metricity of the Vidal connection is put in the next proposition.

Proposition 4.6
Let the triple $(\mathscr{M}, \underline{g}, I)$ denote a riemannian almost product structure with associated metrical decomposition $\underline{g} = g + g'$. Let further $\underline{\tilde{\tilde{\nabla}}}$ denote the Vidal-connection then the following relations hold

$$\begin{aligned}
(\underline{\tilde{\tilde{\nabla}}}_Z g)(X, Y) &= 0 \\
(\underline{\tilde{\tilde{\nabla}}}_{Z'} g)(X, Y) &= -2K_{Z'}(X, Y) = (\mathscr{L}_{Z'} g)(X, Y) \\
(\underline{\tilde{\tilde{\nabla}}}_Z g')(X', Y') &= -2K'_Z(X', Y') = (\mathscr{L}_Z g')(X', Y') \\
(\underline{\tilde{\tilde{\nabla}}}_{Z'} g')(X', Y') &= 0
\end{aligned}$$

In component form the non-metricity tensor can be read off from the next proposition.

Proposition 4.7
Let $\underline{\tilde{Q}}$ be the non-metricity tensor of the Vidal connection, $\underline{\tilde{\tilde{\nabla}}}$, then in component form it reads

$$\begin{aligned}
\underline{\tilde{\tilde{Q}}}_{abc} =& 0 \\
\underline{\tilde{\tilde{Q}}}_{a'bc} =& -2K_{bca'} \\
\underline{\tilde{\tilde{Q}}}_{abc'} =& 0 \\
\underline{\tilde{\tilde{Q}}}_{a'b'c} =& 0 \\
\underline{\tilde{\tilde{Q}}}_{ab'c'} =& -2K'_{b'c'a} \\
\underline{\tilde{\tilde{Q}}}_{a'b'c'} =& 0
\end{aligned}$$

The two traces following from a three tensor symmetric in two indices is found to be

$$\begin{aligned}
\underline{\tilde{\tilde{Q}}}^{1}_{a} =& 0 \\
\underline{\tilde{\tilde{Q}}}^{1}_{a'} =& 0 \\
\underline{\tilde{\tilde{Q}}}_{a} :=& \underline{\tilde{\tilde{Q}}}^{2}_{a} = -2\kappa'_a \\
\underline{\tilde{\tilde{Q}}}_{a'} :=& \underline{\tilde{\tilde{Q}}}^{2}_{a'} = -2\kappa_{a'}
\end{aligned}$$

The basic curvature relations are found from definitions 1 and 4.3, they are given in the proposition below,

Proposition 4.8
Let $\underline{\tilde{\tilde{\nabla}}}$ be the Vidal connection and $\underline{\tilde{\tilde{R}}}$ denote the curvature tensor theorof then the different parts reads

$$
\begin{aligned}
\underline{\tilde{\tilde{R}}}(X,Y)Z &= R^{\mathfrak{D}}(X,Y)Z - 2\mathcal{P}[L(X,Y),Z] \\
\underline{\tilde{\tilde{R}}}(X,Y)Z' &= -2(\underline{\tilde{\tilde{\nabla}}}_{Z'}L)(X,Y) \\
\underline{\tilde{\tilde{R}}}(X,Y')Z &= \mathcal{P}\nabla_X\mathcal{P}[Y',Z] - \mathcal{P}[Y',\mathcal{P}\nabla_X Z] - \mathcal{P}\nabla_{\mathcal{P}[X,Y']}Z - \mathcal{P}[\mathcal{P}'[X,Y'],Z]
\end{aligned}
$$

In order to deal with the curvature identities it is convenient to put these relations in component form.

Proposition 4.9
Let $\underline{\tilde{R}}$ be the Riemann-tensor with respect to the Vidal-connection, then $\underline{\tilde{\tilde{R}}}$ has the following components,

$$
\begin{aligned}
\underline{\tilde{\tilde{R}}}_{abc}{}^{d} =& R^{\mathfrak{D}}_{abc}{}^{d} - 2L_{ab}{}^{e'}C_{e'c}{}^{d} \\
\underline{\tilde{\tilde{R}}}_{abc}{}^{d'} =& 0 \\
\underline{\tilde{\tilde{R}}}_{abc'}{}^{d} =& 0 \\
\underline{\tilde{\tilde{R}}}_{abc'}{}^{d'} =& -2(\underline{\tilde{\tilde{\nabla}}}_{c'}L)_{ab}{}^{d'} \\
\underline{\tilde{\tilde{R}}}_{a'bc}{}^{d} =& E_{a'}[\omega_{bc}{}^{d}] - E_b[C_{a'c}{}^{d}] - C_{a'b}{}^{e}\omega_{ec}{}^{d} - C_{a'b}{}^{e'}C_{e'c}{}^{d} - C_{a'c}{}^{e}\omega_{be}{}^{d} + \omega_{bc}{}^{e}C_{a'e}{}^{d} \\
\underline{\tilde{\tilde{R}}}_{a'bc}{}^{d'} =& 0 \\
\underline{\tilde{\tilde{R}}}_{a'bc'}{}^{d} =& 0 \\
\underline{\tilde{\tilde{R}}}_{a'bc'}{}^{d'} =& E_{a'}[C_{bc'}{}^{d'}] - E_b[\omega_{a'c'}{}^{d'}] - C_{a'b}{}^{e'}\omega_{e'c'}{}^{d'} - C_{a'b}{}^{e}C_{ec'}{}^{d'} - \omega_{a'c'}{}^{e'}C_{be'}{}^{d'} + C_{bc'}{}^{e'}\omega_{a'e'}{}^{d'} \\
\underline{\tilde{\tilde{R}}}_{a'b'c}{}^{d} =& -2(\underline{\tilde{\tilde{\nabla}}}_{c}L)_{a'b'}{}^{d} \\
\underline{\tilde{\tilde{R}}}_{a'b'c}{}^{d'} =& 0 \\
\underline{\tilde{\tilde{R}}}_{a'b'c'}{}^{d} =& 0 \\
\underline{\tilde{\tilde{R}}}_{a'b'c'}{}^{d'} =& R^{\mathfrak{D}'}_{a'b'c'}{}^{d'} - 2L_{a'b'}{}^{e}C_{ec'}{}^{d'}
\end{aligned}
$$

This is the “raw” expressions for the curvature components of the Vidal connection, but after using the identities of the Riemann tensor seen in section 2.4 these will simplify remarkably. There will also appear a couple of identities involving only parts of the deformation tensor. Starting with the second identity it can be seen, using the Tic-Tac-Toe notation introduced in appendix A, that the second identity can be split according to the rigging into 8 irreducible parts, corresponding to the following young tableaux.

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array} \otimes \boxed{} = \left(\begin{array}{|c|}\hline o\\ \hline o\\ \hline o\\ \hline\end{array} \otimes \boxed{o}\right) \oplus \left(\begin{array}{|c|}\hline o\\ \hline o\\ \hline o\\ \hline\end{array} \otimes \boxed{x}\right) \oplus \left(\begin{array}{|c|}\hline o\\ \hline o\\ \hline x\\ \hline\end{array} \otimes \boxed{o}\right) \oplus \left(\begin{array}{|c|}\hline o\\ \hline o\\ \hline x\\ \hline\end{array} \otimes \boxed{x}\right) \oplus$$
$$\left(\begin{array}{|c|}\hline o\\ \hline x\\ \hline x\\ \hline\end{array} \otimes \boxed{o}\right) \oplus \left(\begin{array}{|c|}\hline o\\ \hline x\\ \hline x\\ \hline\end{array} \otimes \boxed{x}\right) \oplus \left(\begin{array}{|c|}\hline x\\ \hline x\\ \hline x\\ \hline\end{array} \otimes \boxed{o}\right) \oplus \left(\begin{array}{|c|}\hline x\\ \hline x\\ \hline x\\ \hline\end{array} \otimes \boxed{x}\right)$$

Here will be listed only the first half of the identities as they of course are symmetric upon changing primes and unprimes.

$$\underline{\tilde{\tilde{R}}}_{[abc]}{}^{d} = 0 \tag{3}$$
$$\underline{\tilde{\tilde{\nabla}}}_{[a}L_{bc]}{}^{d'} = 0 \tag{4}$$
$$\underline{\tilde{\tilde{R}}}_{c'[ab]}d = -L_{ab}{}^{e'}L'_{c'e'}{}^{d} \tag{5}$$
$$\underline{\tilde{\tilde{R}}}_{abc'}{}^{d'} = -2\underline{\tilde{\tilde{\nabla}}}_{c'}L_{ab}{}^{d'} \tag{6}$$

So in conclusion the first identity is structurally inherited by the totally semi-basic part of the Vidal curvature, the second leads to a bianchi identity for the twisting tensor, the third relating the anti-symmetric part of $\underline{\tilde{\tilde{R}}}_{c'[ab]}{}^{d}$ in terms of the twisting tensors and the fourth a faster way of deriving the $\underline{\tilde{\tilde{R}}}_{abc'}{}^{d'}$ component. Thus, the following proposition is proved:

Proposition 4.10
Let $\underline{\tilde{\tilde{\nabla}}}$ be the Vidal connection associated with a almost product structure and L, L' be the respective twisting tensors of the associated distributions then the following Bianchi identities hold,

$$\underline{\tilde{\tilde{\nabla}}}_{[a}L_{bc]}{}^{d'} = 0$$
$$\underline{\tilde{\tilde{\nabla}}}_{[a'}L'_{b'c']}{}^{d} = 0$$

The third identity which is decomposed as,

$$\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline & \\ \hline\end{array} = \left(\begin{array}{|c|}\hline o\\ \hline o\\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline o & o\\ \hline\end{array}\right) \oplus \left(\begin{array}{|c|}\hline o\\ \hline x\\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline o & o\\ \hline\end{array}\right) \oplus \left(\begin{array}{|c|}\hline o\\ \hline o\\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline o & x\\ \hline\end{array}\right) \oplus$$
$$\left(\begin{array}{|c|}\hline o\\ \hline o\\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline x & x\\ \hline\end{array}\right) \oplus \left(\begin{array}{|c|}\hline o\\ \hline x\\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline o & x\\ \hline\end{array}\right) \oplus \left(\begin{array}{|c|}\hline x\\ \hline x\\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline o & o\\ \hline\end{array}\right) \oplus$$
$$\left(\begin{array}{|c|}\hline x\\ \hline x\\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline o & x\\ \hline\end{array}\right) \oplus \left(\begin{array}{|c|}\hline o\\ \hline x\\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline x & x\\ \hline\end{array}\right) \oplus \left(\begin{array}{|c|}\hline x\\ \hline x\\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline x & x\\ \hline\end{array}\right)$$

In this case there are 9 irreducible parts of which five will be listed and the other follows due to symmetry.

$$
\begin{aligned}
\tilde{\tilde{\underline{R}}}_{ab(cd)} &= -2L_{ab}{}^{e'}K_{cde'} && (7)\\
\tilde{\tilde{\underline{R}}}_{a'b(cd)} &= -(\tilde{\tilde{\underline{\nabla}}}_b K)_{cda'} && (8)\\
0 &= 0, && (9)\\
\tilde{\tilde{\underline{R}}}_{ab(c'd')} &= 2(\tilde{\tilde{\underline{\nabla}}}_{[a}K')_{c'd'|b]} && (10)\\
0 &= 0 && (11)
\end{aligned}
$$

The first identity here gives no new information, the second gives yet another part of the $\tilde{\tilde{\underline{R}}}_{a'bcd}$ component, the third and the fifth contain nothing while the fourth together with the original expression for the $\tilde{\tilde{\underline{R}}}_{abc'd'}$ component gives a new non-trivial identity which proves the next proposition.

Proposition 4.11
Let $\tilde{\underline{\nabla}}$ be the Vidal-connection, K, L, K', L' be the second fundamental tensors of a almost product structure then we have the following identities

$$\tilde{\tilde{\underline{\nabla}}}_{[Z}K'_{W]}(X', Y') + \tilde{\tilde{\underline{\nabla}}}_{(X'}L_{Y')}(Z, W) = 0$$
$$\tilde{\tilde{\underline{\nabla}}}_{[Z'}K_{W']}(X, Y) + \tilde{\tilde{\underline{\nabla}}}_{(X}L'_{Y)}(Z', W') = 0$$

In component form the same expressions read

$$(\tilde{\tilde{\underline{\nabla}}}_{[a}K')_{c'd'|b]} + (\tilde{\tilde{\underline{\nabla}}}_{(c'}L)_{ab|d')} = 0$$
$$(\tilde{\tilde{\underline{\nabla}}}_{[a'}K)_{cd|b']} + (\tilde{\tilde{\underline{\nabla}}}_{(c}L')_{a'b'|d)} = 0$$

Contracted the identities read

$$(\tilde{\tilde{\underline{\nabla}}}_{[a}\kappa')_{b]} + (\tilde{\tilde{\underline{\nabla}}}_{c'}L)_{ab}{}^{c'} = 0$$
$$(\tilde{\tilde{\underline{\nabla}}}_{[a'}\kappa)_{b']} + (\tilde{\tilde{\underline{\nabla}}}_{c}L)_{a'b'}{}^{c} = 0$$

Here the notation

$$L_{X'}(Y, Z) := \underline{g}(L(Y, Z), X')$$

is used. The last identity of the Riemann curvature is the most non-trivial of them all. It can be decomposed through the box symmetry, i.e.

$$\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} = \begin{array}{|c|c|}\hline o & o \\ \hline o & o \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline o & o \\ \hline o & x \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline o & o \\ \hline x & x \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline o & x \\ \hline o & x \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline x & x \\ \hline x & o \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline x & x \\ \hline x & x \\ \hline \end{array}.$$

The only identity of these which gives new information is the second why it is the only one listed. This identity though gives the opportunity of writning the entire $\tilde{\tilde{\underline{R}}}_{a'bcd}$ component purely in terms of the different parts of the deformation tensor and its derivatives.

$$\tilde{\tilde{\underline{R}}}_{a'bcd} = -(\tilde{\tilde{\underline{\nabla}}}_b K)_{cda'} - 2(\tilde{\tilde{\underline{\nabla}}}_{[c}K)_{d]ba'} + 2L_{cd}{}^{e'}L'_{a'e'b} - 4L_{b[c}{}^{e'}L'_{a'e'|d]} \qquad (12)$$

This relation together with the identity in 4.11 prooves the following proposition which is the final form of the Vidal curvature components.

Proposition 4.12
Let $\underline{\tilde{\tilde{R}}}$ be the curvature tensor of the Vidal connection then its components can be written

$$
\begin{aligned}
\underline{\tilde{\tilde{R}}}_{abcd} &= R^{\mathcal{D}}_{abcd} - 2L_{ab}{}^{e'}C_{e'cd}\\
\underline{\tilde{\tilde{R}}}_{abcd'} &= 0\\
\underline{\tilde{\tilde{R}}}_{abc'd} &= 0\\
\underline{\tilde{\tilde{R}}}_{abc'd'} &= 2[(\underline{\tilde{\tilde{\nabla}}}_{[a}K')_{c'd'|b]} - (\underline{\tilde{\tilde{\nabla}}}_{[c'}L)_{ab|d']}]\\
\underline{\tilde{\tilde{R}}}_{a'bcd} &= -(\underline{\tilde{\tilde{\nabla}}}_{b}K)_{cda'} - 2(\underline{\tilde{\tilde{\nabla}}}_{[c}K)_{d]ba'} + 2L_{cd}{}^{e'}L'_{a'e'b} - 4L_{b[c}{}^{e'}L'_{a'e'|d]}\\
\underline{\tilde{\tilde{R}}}_{a'bcd'} &= 0\\
\underline{\tilde{\tilde{R}}}_{a'bc'd} &= 0\\
\underline{\tilde{\tilde{R}}}_{a'bc'd'} &= (\underline{\tilde{\tilde{\nabla}}}_{a'}K')_{c'd'b} + 2(\underline{\tilde{\tilde{\nabla}}}_{[c'}K')_{d']a'b} - 2L'_{c'd'}{}^{e}L_{bea'} + 4L'_{a'[c'}{}^{e}L_{be|d']}\\
\underline{\tilde{\tilde{R}}}_{a'b'cd} &= 2[(\underline{\tilde{\tilde{\nabla}}}_{[a'}K)_{cd|b']} - (\underline{\tilde{\tilde{\nabla}}}_{[c}L')_{a'b'|d]}],\\
\underline{\tilde{\tilde{R}}}_{a'b'cd'} &= 0\\
\underline{\tilde{\tilde{R}}}_{a'b'c'd} &= 0\\
\underline{\tilde{\tilde{R}}}_{a'b'c'd'} &= R^{\mathcal{D}'}_{a'b'c'd'} - 2L_{a'b'}{}^{e}C_{ec'd'}
\end{aligned}
$$

From the final expressions of the Vidal curvature components the Schouten two-form and the Ricci tensor can be derived. For the Schouten two-form it is easily seen that it ends up as a total exterior derivative of the two mean curvatures by looking at the trace of the curvature two-form in the Cartan formalism, namely,

$$V := R_{\bar{c}}{}^{\bar{c}} = \underline{d}\tilde{\tilde{\omega}}_{\bar{c}}{}^{\bar{c}} = \underline{d}\kappa + \underline{d}\kappa'. \tag{13}$$

Now of course $\tilde{\tilde{\omega}}$ is only a local object, and therefor it is not sure that V can be written as an exact form globally - this leaves us with the following proposition:

Proposition 4.13
Let $\underline{\tilde{\tilde{\nabla}}}$ be the Vidal connection then the Schouten two-form, $\underline{\tilde{\tilde{V}}}$, of the Vidal connection can locally be written

$$V = d\kappa_I$$

where $\kappa_I = \kappa + \kappa'$. From the integrability condition $dV = 0$ it is obvious that

$$V \in H^2(\underline{\mathscr{M}}),$$

where $H^2(\underline{\mathscr{M}})$ denotes the second cohomology group of the manifold $\underline{\mathscr{M}}$. In component form the Schouten two-form looks like

$$
\begin{aligned}
\underline{\tilde{\tilde{V}}}_{ab} &= 2(\underline{\tilde{\tilde{\nabla}}}_{[a}\kappa')_{b]} - 2L_{ab}{}^{c'}\kappa_{c'},\\
\underline{\tilde{\tilde{V}}}_{a'b} &= (\underline{\tilde{\tilde{\nabla}}}_{a'}\kappa')_{b} - (\underline{\tilde{\tilde{\nabla}}}_{b}\kappa)_{a'},\\
\underline{\tilde{\tilde{V}}}_{a'b'} &= 2(\underline{\tilde{\tilde{\nabla}}}_{[a'}\kappa)_{b']} - 2L_{a'b'}{}^{c}\kappa'_{c}.
\end{aligned}
$$

Now finally the Ricci tensor and the curvature scalar of the Vidal connection can easily be derived.

Proposition 4.14
Let $\underline{\tilde{\tilde{\nabla}}}$ be the Vidal connection then the Ricci tensor reads in component form

$$
\begin{aligned}
\underline{\tilde{\tilde{R}}}_{ab} =& R^{\mathfrak{D}}_{ab} - 2L_{ac}{}^{e'}C_{e'b}{}^{c} \\
\underline{\tilde{\tilde{R}}}_{a'b} =& -(\underline{\tilde{\tilde{\nabla}}}_{b}\kappa)_{a'} + 4L_{bc}{}^{e'}L'_{a'e'}{}^{c} \\
\underline{\tilde{\tilde{R}}}_{ab'} =& -(\underline{\tilde{\tilde{\nabla}}}_{b'}\kappa')_{a} + 4L_{ac}{}^{e'}L'_{b'e'}{}^{c} \\
\underline{\tilde{\tilde{R}}}_{a'b'} =& R^{\mathfrak{D}'}_{a'b'} - 2L'_{a'c'}{}^{e}C_{eb'}{}^{c'}
\end{aligned}
$$

and the Riemann curvature scalar is given by

$$
\underline{\tilde{\tilde{R}}} = \tilde{\tilde{R}} + \tilde{\tilde{R}}' = R^{\mathfrak{D}} + R^{\mathfrak{D}'} - 2L_{ab}{}^{c'}C_{c'}{}^{ab} - 2L'_{a'b'}{}^{c}C_{c}{}^{a'b'}
$$

4.2 The adapted connection

In direct analogy to the previous section, several curvature- and torsion relations are derived with respect to the adapted connection. In contrast to the Vidal-connection the adapted one is metric. In this case the Nijenhuis-tensor is related to the torsion in the way showed by the next proposition. The metricity of the connection has its price though, as is seen below the torsion tensor is more complicated in this case. Some generalized Bianchi identities for the twisting tensor L is also yielded in this case. All tensors, except the totally semi-basic ones, derived from the Riemann tensor, are expressed in terms of the irreducible parts of the deformation tensor.

Proposition 4.15
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ define an riemannian almost product structure, let N_I denote the Nijenhuis tensor of I and $\underline{\tilde{\nabla}}$ the adapted connection defined in 3.17, then we have the following relation,

$$
\frac{1}{2}N_I(X,Y) = \underline{\tilde{T}}(X,Y) + \underline{\tilde{T}}(IX,IY)
$$

Proposition 4.16
Let $\underline{\tilde{T}}$ be the torsion tensor of the adapted connection, $\underline{\tilde{\nabla}}$, then in component

form it reads,

$$
\begin{aligned}
\tilde{\underline{T}}_{ab}{}^{c} =& 0\\
\tilde{\underline{T}}_{ab}{}^{c'} =& -2L_{ab}{}^{c'}\\
\tilde{\underline{T}}_{a'b}{}^{c} =& -H_{b}{}^{c}{}_{a'}\\
\tilde{\underline{T}}_{ab'}{}^{c'} =& -H'_{b'}{}^{c'}{}_{a}\\
\tilde{\underline{T}}_{a'b'}{}^{c} =& -2L'_{a'b'}{}^{c}\\
\tilde{\underline{T}}_{a'b'}{}^{c'} =& 0.
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\underline{T}}_{a} =& -\kappa'_{a}\\
\tilde{\underline{T}}_{a'} =& -\kappa_{a'}
\end{aligned}
$$

The curvature components can further be simplified as shown in the next proposition. Notice that they are expressed in terms of the Vidal connection.

Proposition 4.17
Let $\tilde{\underline{R}}$ be the Riemann-tensor with respect to the adapted connection, then $\tilde{\underline{R}}$ has the following components,

$$
\begin{aligned}
\tilde{\underline{R}}_{abc}{}^{d} =& \tilde{\tilde{\underline{R}}}_{abc}{}^{d} + 2L_{ab}{}^{e'}H_{c}{}^{d}{}_{e'}\\
\tilde{\underline{R}}_{abc}{}^{d'} =& 0\\
\tilde{\underline{R}}_{abc'}{}^{d} =& 0\\
\tilde{\underline{R}}_{abc'}{}^{d'} =& \tilde{\tilde{\underline{R}}}_{abc'}{}^{d'} - 2(\tilde{\tilde{\underline{\nabla}}}_{[a}H')_{c'}{}^{d'}{}_{|b]} - 2H'_{c'}{}^{e'}{}_{[a}H'_{e'}{}^{d'}{}_{|b]}\\
\tilde{\underline{R}}_{a'bc}{}^{d} =& \tilde{\tilde{\underline{R}}}_{a'bc}{}^{d} + (\tilde{\tilde{\underline{\nabla}}}_{b}H)_{c}{}^{d}{}_{a'}\\
\tilde{\underline{R}}_{a'bc}{}^{d'} =& 0\\
\tilde{\underline{R}}_{a'bc'}{}^{d} =& 0\\
\tilde{\underline{R}}_{a'bc'}{}^{d'} =& \tilde{\tilde{\underline{R}}}_{a'bc'}{}^{d'} - (\tilde{\tilde{\underline{\nabla}}}_{a'}H)_{c'}{}^{d'}{}_{b}\\
\tilde{\underline{R}}_{a'b'c}{}^{d} =& \tilde{\tilde{\underline{R}}}_{a'b'c}{}^{d} - 2(\tilde{\tilde{\underline{\nabla}}}_{[a'}H)_{c}{}^{d}{}_{|b']} - 2H_{c}{}^{e}{}_{[a'}H'_{e}{}^{d}{}_{|b']}\\
\tilde{\underline{R}}_{a'b'c}{}^{d'} =& 0\\
\tilde{\underline{R}}_{a'b'c'}{}^{d} =& 0\\
\tilde{\underline{R}}_{a'b'c'}{}^{d'} =& \tilde{\tilde{\underline{R}}}_{a'b'c'}{}^{d'} + 2L_{a'b'}{}^{e}H_{c'}{}^{d'}{}_{e}
\end{aligned}
$$

where,

$$
\begin{aligned}
\tilde{\tilde{\underline{R}}}_{abc'}{}^{d'} :=& E_{a}[\Omega_{bc'}{}^{d'}] - E_{b}[\Omega_{ac'}{}^{d'}] - C_{ab}{}^{e}\Omega_{ec'}{}^{d'} - 2\Omega_{[a|c'}{}^{e'}\Omega_{b]e'}{}^{d'} - 2L_{ab}{}^{e'}\omega_{e'c'}{}^{d'}\\
\tilde{\tilde{\underline{R}}}_{a'b'c}{}^{d} :=& E_{a'}[\Omega_{b'c}{}^{d}] + E_{b'}[\Omega_{a'c}{}^{d}] - C_{a'b'}{}^{e'}\Omega_{e'c}{}^{d} - 2\Omega_{[a'|c}{}^{e}\Omega_{b']e}{}^{d} - 2L'_{a'b'}{}^{e}\omega_{ec}{}^{d}
\end{aligned}
$$

Using proposition 4.12 the curvature tensor of the adapted connection can be written entirely in terms of the semi-basic parts of the Vidal curvature, parts of the deformation tensors and the Vidal covariant derivative thereof.

Proposition 4.18
Let $\underline{\tilde{R}}$ be the curvature tensor of the adapted connection then its components can be written

$$
\begin{aligned}
\underline{\tilde{R}}_{abcd} &= \underline{\tilde{\tilde{R}}}_{abcd} + 2L_{ab}{}^{e'}H_{cde'} \\
\underline{\tilde{R}}_{abcd'} &= 0 \\
\underline{\tilde{R}}_{abc'd} &= 0 \\
\underline{\tilde{R}}_{abc'd'} &= -2[(\underline{\tilde{\tilde{\nabla}}}_{[a}L')_{c'd'|b]} + (\underline{\tilde{\tilde{\nabla}}}_{[c'}L)_{ab|d']} - H'_{c'}{}^{e'}{}_{[a}H'_{d'e'|b]}] \\
\underline{\tilde{R}}_{a'bcd} &= (\underline{\tilde{\tilde{\nabla}}}_{b}L)_{cda'} - 2(\underline{\tilde{\tilde{\nabla}}}_{[c}K)_{d]ba'} + 2L_{cd}{}^{e'}L'_{a'e'b} - 4L_{b[c}{}^{e'}L'_{a'e'|d]} \\
\underline{\tilde{R}}_{a'bcd'} &= 0 \\
\underline{\tilde{R}}_{a'bc'd} &= 0 \\
\underline{\tilde{R}}_{a'bc'd'} &= -(\underline{\tilde{\tilde{\nabla}}}_{a'}L')_{c'd'b} + 2(\underline{\tilde{\tilde{\nabla}}}_{[c'}K')_{d']a'b} - 2L'_{c'd'}{}^{e}L_{bea'} + 4L'_{a'[c'}{}^{e}L_{be|d']} \\
\underline{\tilde{R}}_{a'b'cd} &= -2[(\underline{\tilde{\tilde{\nabla}}}_{[a'}L)_{cd|b']} + (\underline{\tilde{\tilde{\nabla}}}_{[c}L')_{a'b'|d]} - H_{c}{}^{e}{}_{[a'}H_{de|b']}] \\
\underline{\tilde{R}}_{a'b'cd'} &= 0 \\
\underline{\tilde{R}}_{a'b'c'd} &= 0 \\
\underline{\tilde{R}}_{a'b'c'd'} &= \underline{\tilde{\tilde{R}}}_{a'b'c'd'} + 2L_{a'b'}{}^{e}H'_{c'd'e}
\end{aligned}
$$

These relations can be expressed purely in terms of the adapted connection instead of the Vidal connection.

Proposition 4.19
Let $\underline{\tilde{R}}$ be the curvature tensor of the adapted connection then its components

can be written

$$
\begin{aligned}
\underline{\tilde{R}}_{abcd} =& \underline{\tilde{\tilde{R}}}_{abcd} + 2L_{ab}{}^{e'}H_{cde'}\\
\underline{\tilde{R}}_{abcd'} =& 0\\
\underline{\tilde{R}}_{abc'd} =& 0\\
\underline{\tilde{R}}_{abc'd'} =& -2[(\underline{\tilde{\nabla}}_{[a}L')_{c'd'|b]} + (\underline{\tilde{\nabla}}_{[c'}L)_{ab|d']} - W'_{c'}{}^{e'}{}_{[a}W'_{d'e'|b]} + L'_{c'}{}^{e'}{}_{[a}L'_{d'e'|b]} +\\
&+ 2L_{[a|}{}^{e}{}_{c'}L_{b]ed'} + 2W_{[a}{}^{e}{}_{[c'}L_{b]e|d']} + \frac{2}{k}L_{ab[c'}\kappa_{d']}]\\
\underline{\tilde{R}}_{a'bcd} =& (\underline{\tilde{\nabla}}_{b}L)_{cda'} - 2(\underline{\tilde{\nabla}}_{[c}K)_{d]ba'} - L_{cd}{}^{e'}H'_{e'a'b} - 2K_{b[c|}{}^{e'}H'_{e'a'|d]}\\
\underline{\tilde{R}}_{a'bcd'} =& 0\\
\underline{\tilde{R}}_{a'bc'd} =& 0\\
\underline{\tilde{R}}_{a'bc'd'} =& -(\underline{\tilde{\nabla}}_{a'}L')_{c'd'b} + 2(\underline{\tilde{\nabla}}_{[c'}K')_{d']a'b} + L_{c'd'}{}^{e}H'_{eba'} + 2K_{a'[c'|}{}^{e}H'_{eb|d']}\\
\underline{\tilde{R}}_{a'b'cd} =& -2[(\underline{\tilde{\nabla}}_{[a'}L)_{cd|b']} + (\underline{\tilde{\nabla}}_{[c}L')_{a'b'|d]} - W_{c}{}^{e}{}_{[a'}W_{de|b']} + L_{c}{}^{e}{}_{[a'}L_{de|b']} +\\
&+ 2L'_{[a'|}{}^{e'}{}_{c}L'_{b']e'd} + 2W'_{[a'}{}^{e'}{}_{[c}L'_{b']e'|d]} + \frac{2}{k'}L'_{a'b'[c}\kappa'_{d]}]\\
\underline{\tilde{R}}_{a'b'cd'} =& 0\\
\underline{\tilde{R}}_{a'b'c'd} =& 0\\
\underline{\tilde{R}}_{a'b'c'd'} =& \underline{\tilde{\tilde{R}}}_{a'b'c'd'} + 2L_{a'b'}{}^{e}H'_{c'd'e}
\end{aligned}
$$

From previous proposition the Ricci tensor of the adpted connection can simply be deduced by contraction as the adapted connection is metric. It should be stressed that this Ricci tensor is in general not symmetric.

Proposition 4.20
Let $\underline{\tilde{\nabla}}$ be the adapted connection then the Ricci tensor reads in component form

$$
\begin{aligned}
\underline{\tilde{R}}_{ab} =& \underline{\tilde{\tilde{R}}}_{ab} + 2L_{ac}{}^{e'}H_{b}{}^{c}{}_{e'}\\
\underline{\tilde{R}}_{a'b} =& (\underline{\tilde{\nabla}}_{c}H)_{b}{}^{c}{}_{a'} - H_{bc}{}^{e'}H'_{e'a'}{}^{c} - (\underline{\tilde{\nabla}}_{b}\kappa)_{a'} + \kappa^{e'}H'_{e'a'b}\\
\underline{\tilde{R}}_{ab'} =& (\underline{\tilde{\nabla}}_{c'}H')_{b'}{}^{c'}{}_{a} - H'_{b'c'}{}^{e}H_{ea}{}^{c'} - (\underline{\tilde{\nabla}}_{b'}\kappa)_{a} + \kappa'^{e}H_{eab'}\\
\underline{\tilde{R}}_{a'b'} =& \underline{\tilde{\tilde{R}}}_{a'b'} + 2L'_{a'c'}{}^{e}H'_{b'}{}^{c'}{}_{e}
\end{aligned}
$$

and the Riemann curvature scalar is given by

$$\underline{\tilde{R}} = \tilde{R} + \tilde{R}' = \tilde{\tilde{R}} + \tilde{\tilde{R}}' + 2L_{ab}{}^{c'}L^{ab}{}_{c'} + 2L'_{a'b'}{}^{c}L'^{a'b'}{}_{c}$$

The generalized Biachi-identities for the twisting tensor L are given in next proposition. They were derived by using the antisymmetry ((i) of proposition 2.4) and the components expressed in terms of the adapted connection, see proposition 4.19.

Proposition 4.21
Let $\underline{\tilde{\nabla}}$ be the adapted connection associated with an almost product structure and L, L', K, K' be the respective twisting and extrinsic curvature tensors of the associated distributions then the following identities hold

$$\begin{aligned}\underline{\tilde{\nabla}}_{[a}L_{bc]}{}^{d'} + L_{[ab|}{}^{e'}H_{e'}{}^{d'}{}_{|c]} =&0\\ \underline{\tilde{\nabla}}_{[a'}L_{b'c']}{}^{d} + L_{[a'b'|}{}^{e}H_{e}{}^{d}{}_{|c']} =&0\end{aligned}$$

As was seen in previous subsection, where the Vidal connection was studied, new identities between parts of the deformation tensors arose as integrability conditions on these while imposing the identities of the curvature tensor. These were derived in proposition 4.19 above and in the same fashion the corresponding identities for the adapted connection arise.

Proposition 4.22
Let $\underline{\tilde{\nabla}}$ be the adapted connection, K, L, K', L' be the second fundamental tensors with respect to a almost product structure, I, then the following identities hold

$$\begin{aligned}(\underline{\tilde{\nabla}}_{[a}K')_{c'd'|b]} + (\underline{\tilde{\nabla}}_{(c'}L)_{ab|d')} - 2L'_{(c'|}{}^{e'}{}_{[a}K'_{e'|d')|b]} - 2K_{[a|}{}^{e}{}_{(c'}L_{e|b]d')} =&0,\\ (\underline{\tilde{\nabla}}_{[a'}K)_{cd|b']} + (\underline{\tilde{\nabla}}_{(c}L')_{a'b'|d)} - 2L_{(c|}{}^{e}{}_{[a'}K_{e|d)|b']} - 2K'_{[a'|}{}^{e'}{}_{(c}L'_{e'|b']d)} =&0.\end{aligned}$$

These identities look in the contracted case like

$$\begin{aligned}(\underline{\tilde{\nabla}}_{[a}\kappa')_{b]} + (\underline{\tilde{\nabla}}_{c'}L)_{ab}{}^{c'} - 2W_{[a|}{}^{e}{}_{c'}L_{e|b]}{}^{c'} - \frac{2}{k}L_{ab}{}^{c'}\kappa_{c'} =&0,\\ (\underline{\tilde{\nabla}}_{[a'}\kappa)_{b']} + (\underline{\tilde{\nabla}}_{c}L)_{a'b'}{}^{c} - 2W_{[a'|}{}^{e'}{}_{c}L_{e'|b']}{}^{c} - \frac{2}{k'}L_{a'b'}{}^{c}\kappa'_{c} =&0.\end{aligned}$$

4.3 The Levi-Civita connection

In this section all curvature relations for the Levi-Civita connection is given. The curvature, Ricci and the curvature scalar are expressed in terms of the irreducible components of the deformation-tensor. Starting from Cartan's structure equations and writing the curvature two-form as,

$$\begin{aligned}\underline{R}_{\bar{c}}{}^{\bar{d}} :=& \underline{d\omega}_{\bar{c}}{}^{\bar{d}} - \underline{\omega}_{\bar{c}}{}^{\bar{e}}\wedge\underline{\omega}_{\bar{e}}{}^{\bar{d}} =\\ =&\begin{pmatrix} \underline{d\omega}_{c}{}^{d} - \underline{\omega}_{c}{}^{e}\wedge\underline{\omega}_{e}{}^{d} - \underline{H}_{c}{}^{e'}\wedge\underline{H}_{e'}{}^{d}, & \underline{dH}_{c}{}^{d'} - \underline{\omega}_{c}{}^{e}\wedge\underline{H}_{e}{}^{d'} - H_{c}{}^{e'}\wedge\underline{\omega}_{e'}{}^{d'}\\ \underline{dH}_{c'}{}^{d} - \underline{\omega}_{c'}{}^{e'}\wedge\underline{H}_{e'}{}^{d} - H_{c'}{}^{e}\wedge\underline{\omega}_{e}{}^{d}, & \underline{d\omega}_{c'}{}^{d'} - \underline{\omega}_{c'}{}^{e'}\wedge\underline{\omega}_{e'}{}^{d'} - \underline{H}_{c'}{}^{e}\wedge\underline{H}_{e}{}^{d'}\end{pmatrix}\end{aligned}$$

the components can be given in terms of the adapted connection.

Proposition 4.23
Let $\underline{R}$ be the Riemann-tensor with respect to the Levi-Civita connection, then

$\underline{R}$ has the following components,

$$
\begin{aligned}
\underline{R}_{abc}{}^{d} &= \underline{\tilde{R}}_{abc}{}^{d} + 2H_{[a|c}{}^{e'}H_{b]}{}^{d}{}_{e'}\\
\underline{R}_{abc}{}^{d'} &= 2(\underline{\tilde{\nabla}}_{[a}H)_{b]c}{}^{d'} + 2L_{ab}{}^{e'}H'_{e'}{}^{d'}{}_{c}\\
\underline{R}_{abc'}{}^{d} &= -2(\underline{\tilde{\nabla}}_{[a}H)_{b]}{}^{d}{}_{c'} - 2L_{ab}{}^{e'}H'_{e'c'}{}^{d}\\
\underline{R}_{abc'}{}^{d'} &= \underline{\tilde{R}}_{abc'}{}^{d'} + 2H_{[a|}{}^{e}{}_{c'}H_{b]e}{}^{d'}\\
\underline{R}_{a'bc}{}^{d} &= \underline{\tilde{R}}_{a'bc}{}^{d} - H'_{a'}{}^{e'}{}_{c}H_{b}{}^{d}{}_{e'} + H_{bc}{}^{e'}H'_{a'e'}{}^{d}\\
\underline{R}_{a'bc}{}^{d'} &= (\underline{\tilde{\nabla}}_{a'}H)_{bc}{}^{d'} + (\underline{\tilde{\nabla}}_{b}H')_{a'}{}^{d'}{}_{c} - H_{b}{}^{e}{}_{a'}H_{ec}{}^{d'} - H'_{a'}{}^{e'}{}_{b}H'_{e'}{}^{d'}{}_{c}\\
\underline{R}_{a'bc'}{}^{d} &= -(\underline{\tilde{\nabla}}_{a'}H)_{b}{}^{d}{}_{c'} - (\underline{\tilde{\nabla}}_{b}H')_{a'c'}{}^{d} + H_{b}{}^{e}{}_{a'}H_{e}{}^{d}{}_{c'} + H'_{a'}{}^{e'}{}_{b}H'_{e'c'}{}^{d}\\
\underline{R}_{a'bc'}{}^{d'} &= \underline{\tilde{R}}_{a'bc'}{}^{d'} - H'_{a'c'}{}^{e}H_{be}{}^{d'} + H_{b}{}^{e}{}_{c'}H'_{a'}{}^{d'}{}_{e}\\
\underline{R}_{a'b'c}{}^{d} &= \underline{\tilde{R}}_{a'b'c}{}^{d} + 2H'_{[a'|}{}^{e'}{}_{c}H'_{b']e'}{}^{d}\\
\underline{R}_{a'b'c}{}^{d'} &= -2(\underline{\tilde{\nabla}}_{[a'}H')_{b']}{}^{d'}{}_{c} - 2L'_{a'b'}{}^{e}H_{ec}{}^{d'}\\
\underline{R}_{a'b'c'}{}^{d} &= 2(\underline{\tilde{\nabla}}_{[a'}H')_{b']c'}{}^{d} + 2L'_{a'b'}{}^{e}H_{e}{}^{d}{}_{c'}\\
\underline{R}_{a'b'c'}{}^{d'} &= \underline{\tilde{R}}_{a'b'c'}{}^{d'} + 2H'_{[a'|c'}{}^{e}H'_{b']}{}^{d'}{}_{e}
\end{aligned}
$$

These are also known as the Gauss-Codazzi relations. In the case of the Levi–Civita connection though, it is clear that its curvature tensor possesses the box symmetry, i.e.

$$
\begin{array}{|c|c|}
\hline
 & \\
\hline
 & \\
\hline
\end{array}
$$

From the Tic–Tac–Toe notation it follows that the box symmetry reduces to six irreducible parts under a rigging. From the previous analysis of the Vidal and the adapted curvatures these can again be written entirely in terms of the semi-basic components and parts of the deformation tensors.

Proposition 4.24
Let $\underline{R}$ be the Riemann tensor with respect to the Levi–Civita connection then its components can be written in terms of just the deformation tensor and adapted covariant derivatives thereof plus the complete longitudal and normal parts of

the adapted curvature,

$$
\begin{aligned}
\underline{R}_{abcd} =& \underline{\tilde{R}}_{abcd} + 2H_{[a|c}{}^{e'}H_{b]de'}\\
\underline{R}_{abcd'} =& 2(\underline{\tilde{\nabla}}_{[a}H)_{b]cd'} + 2L_{ab}{}^{e'}H'_{e'd'c}\\
\underline{R}_{abc'd'} =& -2[(\underline{\tilde{\nabla}}_{[c'}L)_{ab|d']} + (\underline{\tilde{\nabla}}_{[a}L')_{c'd'|b]} + L_{[a|}{}^{e}{}_{c'}L_{b]ed'} + L'_{[c'|}{}^{e'}{}_{a}L'_{d']e'b} -\\
& - W_{[a|}{}^{e}{}_{c'}W_{b]ed'} - W'_{[c'|}{}^{e'}{}_{a}W'_{d']e'b}]\\
\underline{R}_{a'bc'd} =& \frac{1}{2}\underline{R}_{a'c'bd} - (\underline{\tilde{\nabla}}_{(a'|}K)_{bd|c')} - (\underline{\tilde{\nabla}}_{(b}K')_{a'c'|d)} + K_{(b|}{}^{e}{}_{a'}K_{d)ec'} - L_{(b|}{}^{e}{}_{a'}L_{d)ec'} +\\
& + K'_{(a'|}{}^{e'}{}_{b}K'_{c)e'd} - L'_{(a'|}{}^{e'}{}_{b}L'_{c)e'd}\\
\underline{R}_{a'b'c'd} =& 2(\underline{\tilde{\nabla}}_{[a'}H')_{b']c'd} + 2L'_{a'b'}{}^{e}H_{edc'}\\
\underline{R}_{a'b'c'd'} =& \underline{\tilde{R}}_{a'b'c'd'} + 2H'_{[a'|c'}{}^{e}H'_{b']d'e}
\end{aligned}
$$

In ref. [5] these were deduced in terms of the Levi–Civita connection but as it does not preserve the rigging these are better expressed in terms of the adapted or the Vidal connection. Note that the expressions are only decomposed in terms of the irreducible parts of the deformation tensors, where it is necessary in order to make manifest the symmetries. In all other cases it is a straight forward process to do just by insertion.

Proposition 4.25
Let $\underline{R}$ be the Riemann tensor with respect to the Levi–Civita connection then its components can be written in terms of just the deformation tensor and Vidal covariant derivatives thereof plus the complete longitudal and normal parts of the Vidal curvature as

$$
\begin{aligned}
\underline{R}_{abcd} =& \underline{\tilde{\tilde{R}}}_{abcd} + 2L_{ab}{}^{e'}H_{cde'} + 2H_{[a|c}{}^{e'}H_{b]de'}\\
\underline{R}_{abcd'} =& 2(\underline{\tilde{\tilde{\nabla}}}_{[a}H)_{b]cd'} + 2H_{[b|c}{}^{e'}H'_{d'e'|a]} + 2L_{ab}{}^{e'}H'_{e'd'c}\\
\underline{R}_{abc'd'} =& -2[(\underline{\tilde{\tilde{\nabla}}}_{[c'}L)_{ab|d']} + (\underline{\tilde{\tilde{\nabla}}}_{[a}L')_{c'd'|b]} + \frac{2}{k'}\kappa'_{[a}L'_{c'd'|b]} + \frac{2}{k}\kappa_{[c'}L_{ab|d']}\\
& - W_{[a|}{}^{e}{}_{c'}W_{b]ed'} - W'_{[c'|}{}^{e'}{}_{a}W'_{d']e'b} - 2W_{[a|}{}^{e}{}_{[c'}L_{b]e|d']} - 2W'_{[c'|}{}^{e'}{}_{[a}L'_{d']e'|b]} -\\
& - L_{[a|}{}^{e}{}_{c'}L_{b]ed'} - L'_{[c'|}{}^{e'}{}_{a}L'_{d']e'b}]\\
\underline{R}_{a'bc'd} =& \frac{1}{2}\underline{R}_{bda'c'} - \underline{\tilde{\tilde{\nabla}}}_{(a'|}K_{bd|c')} - \underline{\tilde{\tilde{\nabla}}}_{(b|}K'_{a'c'|d)} - W_{(d|}{}^{e}{}_{(a'}W_{b)ec')} - L_{(d|}{}^{e}{}_{(a'}L_{b)ec')} -\\
& - 2W_{(d}{}^{e}{}_{(a'|}L_{b)e|c')} - \frac{1}{k^2}\kappa_{a'}\kappa_{c'}\eta_{bd} - \frac{1}{k}\kappa_{(a'|}W_{bd|c')} -\\
& - W'_{(c'|}{}^{e'}{}_{(b|}W'_{a')e'd)} - L'_{(c'|}{}^{e'}{}_{(b}L'_{d)e'a')} -\\
& - 2W'_{(c'}{}^{e'}{}_{(b|}L'_{d)e'|a')} - \frac{1}{k'^2}\kappa'_{b}\kappa'_{d}\eta_{a'c'} - \frac{1}{k'}\kappa'_{(b|}W_{a'c'|d)}\\
\underline{R}_{a'b'c'd} =& 2(\underline{\tilde{\tilde{\nabla}}}_{[a'}H')_{b']c'd} + 2H'_{[b'|c'}{}^{e}H_{de|a']} + 2L'_{a'b'}{}^{e}H_{edc'}\\
\underline{R}_{a'b'c'd'} =& \underline{\tilde{\tilde{R}}}_{a'b'c'd'} + 2L'_{a'b'}{}^{e}H'_{c'd'e} + 2H'_{[a'|c'}{}^{e}H'_{b']d'e}
\end{aligned}
$$

From propositions 4.24 and 4.25 the Ricci tensor and the curvature scalar is most easily deduced.

Proposition 4.26
Let $\underline{\nabla}$ be the Levi-Civita connection then the Ricci tensor reads in terms of the adapted connection,

$$\begin{aligned}
\underline{R}_{ab} =& R_{ab} + R''_{ab} = \underline{\tilde{R}}_{(ab)} - \frac{1}{k}(\underline{\tilde{\nabla}}_{c'}\kappa)^{c'}\eta_{ab} - (\underline{\tilde{\nabla}}_{(a}\kappa')_{b)} - (\underline{\tilde{\nabla}}_{c'}W)_{ab}{}^{c'} + \frac{1}{k}\kappa^2\eta_{ab} + \\
& + W_{ab}{}^{c'}\kappa_{c'} + \frac{1}{k'}\kappa'{}_a\kappa'{}_b + W'_{c'e'a}W'^{c'e'}{}_b - L'_{c'e'a}L'^{c'e'}{}_b \\
\underline{R}_{ab'} =& (\frac{1-k}{k})(\underline{\tilde{\nabla}}_a\kappa)_{b'} + (\frac{1-k'}{k'})(\underline{\tilde{\nabla}}_{b'}\kappa')_a + (\underline{\tilde{\nabla}}_c W)_a{}^c{}_{b'} + \\
& + (\underline{\tilde{\nabla}}_{c'}W')_{b'}{}^{c'}{}_a + (\underline{\tilde{\nabla}}_c L)_a{}^c{}_{b'} + (\underline{\tilde{\nabla}}_{c'}L')_{b'}{}^{c'}{}_a + \\
& + 4L_a{}^{ec'}L'_{b'c'e} - 2L_a{}^{ce'}W'{}_{b'e'c} - 2L'_{b'}{}^{c'e}W_{aec'} - \frac{2}{k'}L_{acb'}\kappa'^c - \frac{2}{k}L_{b'c'a}\kappa^{c'}, \\
\underline{R}_{a'b'} =& R'_{a'b'} + R''_{a'b'} = \underline{\tilde{R}}_{(a'b')} - \frac{1}{k'}(\underline{\tilde{\nabla}}_c\kappa')^c\eta'{}_{a'b'} - (\underline{\tilde{\nabla}}_{(a'}\kappa)_{b')} - (\underline{\tilde{\nabla}}_c W')_{a'b'}{}^c + \frac{1}{k'}\kappa'^2\eta'{}_{a'b'} + \\
& + W'_{a'b'}{}^c\kappa'_c + \frac{1}{k}\kappa_{a'}\kappa_{b'} + W_{cea'}W^{ce}{}_{b'} - L_{cea'}L^{ce}{}_{b'}
\end{aligned}$$

where the following definitions are used

$$\begin{aligned}
R_{ab} :=& \underline{R}_{acb}{}^c = \underline{\tilde{R}}_{(ab)} - W_a{}^{ec'}W_{bec'} + (\frac{k-2}{k})\kappa^{c'}W_{abc'} + (\frac{k-1}{k^2})\eta_{ab}\kappa^2 + L_a{}^{ec'}L_{bec'} \\
R''_{ab} :=& \underline{R}_{ac'b}{}^{c'} = -(\underline{\tilde{\nabla}}_{(a}\kappa')_{b)} - (\underline{\tilde{\nabla}}_{c'}W)_{ab}{}^{c'} - \frac{1}{k}(\underline{\tilde{\nabla}}_{c'}\kappa)^{c'}\eta_{ab} + W'^{c'e'}{}_a W'{}_{c'e'b} + \frac{1}{k'}\kappa'_a\kappa'_b - \\
& - L'^{c'e'}{}_a L'_{c'e'b} + W_a{}^{ec'}W_{bec'} + \frac{2}{k}\kappa^{c'}W_{abc'} + \frac{1}{k^2}\eta_{ab}\kappa^2 - L_a{}^{ec'}L_{bec'}
\end{aligned}$$

and similarly for the $R'_{a'b'}$ component. The Riemann curvature scalar is given by

$$\underline{R} = R + 2R'' + R' = \tilde{R} + \tilde{R}' + \frac{1-k}{k}\kappa^2 + \frac{1-k'}{k'}\kappa'^2 - 2\underline{\nabla}\cdot\kappa_I + W^2 + W'^2 - L^2 - L'^2$$

where the following definitions are used

$$\begin{aligned}
R :=& \underline{R}_{ab}{}^{ab} = \tilde{R} + \kappa^2 - K^2 + L^2 \\
R'' :=& \underline{R}_{ab'}{}^{ab'} = -\underline{\tilde{\nabla}}_{a'}\kappa^{a'} - \underline{\tilde{\nabla}}_a\kappa'^a + W^2 - L^2 + \frac{1}{k}\kappa^2 + W'^2 - L'^2 + \frac{1}{k'}\kappa'^2 \\
R' :=& \underline{R}_{a'b'}{}^{a'b'} = \tilde{R}' + \kappa'^2 - K'^2 + L'^2
\end{aligned}$$

Part of which is found in [7]. In the expressions on the curvature scalar above, we have used the following relation

$$\underline{\nabla}\cdot\kappa_I = (\underline{\tilde{\nabla}}_{a'}\kappa)^{a'} + (\underline{\tilde{\nabla}}_a\kappa')^a - \kappa^2 - \kappa'^2 \tag{14}$$

(15)

and the notation $L^2 := L_{abc'}L^{abc'}$ etc. In terms of the Vidal connection the Ricci tensor and the curvature scalar is given in next proposition.

Proposition 4.27
Let $\underline{\nabla}$ be the Levi-Civita connection then the Ricci tensor reads, in component form, in terms of the Vidal connection

$$
\begin{aligned}
\underline{R}_{ab} =& \underline{\tilde{R}}_{(ab)} - \frac{1}{k}(\underline{\tilde{\tilde{\nabla}}}_{c'}\kappa)^{c'}\eta_{ab} - (\underline{\tilde{\tilde{\nabla}}}_{(a}\kappa')_{b)} - (\underline{\tilde{\tilde{\nabla}}}_{c'}W)_{ab}{}^{c'} - 2W_a{}^{ec'}W_{ebc'} + \\
&+ (\frac{k-2}{k})W_{ab}{}^{c'}\kappa_{c'} + \frac{1}{k'}\kappa'{}_a\kappa'{}_b + \frac{1}{k}\kappa^2\eta_{ab} + W'_{c'e'a}W'^{c'e'}{}_b - \\
&- L'_{c'e'a}L'^{c'e'}{}_b + 2L_a{}^{ec'}L_{bec'} \\
\underline{R}_{ab'} =& (\frac{1-k}{k})(\underline{\tilde{\tilde{\nabla}}}_a\kappa)_{b'} + (\frac{1-k'}{k'})(\underline{\tilde{\tilde{\nabla}}}_{b'}\kappa')_a + (\underline{\tilde{\tilde{\nabla}}}_c W)_a{}^c{}_{b'} + \\
&+ (\underline{\tilde{\tilde{\nabla}}}_{c'}W')_{b'}{}^{c'}{}_a + (\underline{\tilde{\tilde{\nabla}}}_c L)_a{}^c{}_{b'} + (\underline{\tilde{\tilde{\nabla}}}_{c'}L')_{b'}{}^{c'}{}_a + \\
&+ 6L_a{}^{ec'}L'_{b'c'e} + 2W_a{}^{ec'}W'_{b'c'e} - (\frac{k+k'-2}{kk'})\kappa'_a\kappa_{b'} - \\
&- L_{acb'}\kappa'^c - L_{b'c'a}\kappa^{c'} - (\frac{k'-2}{k'})W_{acb'}\kappa'^c - (\frac{k-2}{k})W_{b'c'a}\kappa^{c'} \\
\underline{R}_{a'b'} =& \underline{\tilde{R}}_{(a'b')} - \frac{1}{k'}(\underline{\tilde{\tilde{\nabla}}}_c\kappa')^c\eta'{}_{a'b'} - (\underline{\tilde{\tilde{\nabla}}}_{(a'}\kappa)_{b')} - (\underline{\tilde{\tilde{\nabla}}}_c W')_{a'b'}{}^c 2W'_{a'}{}^{e'c}W'_{e'b'c} - \\
&+ (\frac{k'-2}{k'})W'_{a'b'}{}^c\kappa'_c + \frac{1}{k}\kappa_{a'}\kappa_{b'} + \frac{1}{k'}\kappa'^2\eta'{}_{a'b'} + W_{cea'}W^{ce}{}_{b'} - \\
&- L_{cea'}L^{ce}{}_{b'} + 2L'_{a'}{}^{e'c}L'_{e'b'c}
\end{aligned}
$$

where the following definitions were used

$$
\begin{aligned}
R_{ab} :=& \underline{R}_{acb}{}^c = \underline{\tilde{\tilde{R}}}_{(ab)} - W_a{}^{ec'}W_{bec'} + (\frac{k-2}{k})\kappa^{c'}W_{abc'} + (\frac{k-1}{k^2})\eta_{ab}\kappa^2 \\
&+ 3L_a{}^{ec'}L_{bec'} + 2L_{(a|}{}^{ce'}W_{b)ce'} \\
R''_{ab} :=& \underline{R}_{ac'b}{}^{c'} = -(\underline{\tilde{\tilde{\nabla}}}_{(a}\kappa')_{b)} - (\underline{\tilde{\tilde{\nabla}}}_{c'}W)_{ab}{}^{c'} - \frac{1}{k}(\underline{\tilde{\tilde{\nabla}}}_{c'}\kappa)^{c'}\eta_{ab} + W'^{c'e'}{}_aW'{}_{c'e'b} + \frac{1}{k'}\kappa'_a\kappa'_b - \\
&- L'^{c'e'}{}_aL'_{c'e'b} - W_a{}^{ec'}W_{bec'} + \frac{1}{k^2}\eta_{ab}\kappa^2 - L_a{}^{ec'}L_{bec'} - 2L_{(a}{}^{ec'}W_{e|b)c'}
\end{aligned}
$$

again similarly for the $R'_{a'b'}$ component. The Riemann curvature scalar is given by

$$
\underline{R} = R + 2R'' + R' = \tilde{\tilde{R}} + \tilde{\tilde{R}}' + \frac{1-k}{k}\kappa^2 + \frac{1-k'}{k'}\kappa'^2 - 2\underline{\nabla}\cdot\kappa_I + W^2 + W'^2 + L^2 + L'^2
$$

where the following definitions were used

$$
\begin{aligned}
R =& \tilde{\tilde{R}} + \kappa^2 - K^2 + 3L^2 \\
R'' =& -\underline{\tilde{\tilde{\nabla}}}_{a'}\kappa^{a'} - \underline{\tilde{\tilde{\nabla}}}_a\kappa'^a + W^2 - L^2 + \frac{1}{k}\kappa^2 + W'^2 - L'^2 + \frac{1}{k'}\kappa'^2 \\
R' =& \tilde{\tilde{R}}' + \kappa'^2 - K'^2 + 3L'^2
\end{aligned}
$$

4.4 Their conformal properties

In an earlier treatment it was found that conformal transformations does affect a connection including torsion and non-metricity in a non-trivial fashion. Here will be given a complete analysis of the induced transformations of the Vidal, adapted and Levi–Civita connections.

Definition 4.28
Let the triplet $(\underline{\mathscr{M}}, \underline{g}, I)$ *denote an almost product manifold, let* $\underline{\nabla}, \underline{\tilde{\nabla}}, \underline{\tilde{\tilde{\nabla}}}$ *denote the Levi–Civita, adapted and Vidal connection respectively then define the associated* **conformal tensors** *denoted* $\underline{\mathscr{C}}, \underline{\tilde{\mathscr{C}}}, \underline{\tilde{\tilde{\mathscr{C}}}}$ *with characteristics*

$$\underline{\mathscr{C}}, \underline{\tilde{\mathscr{C}}}, \underline{\tilde{\tilde{\mathscr{C}}}}: \qquad \Lambda^1 \times \Lambda^1 \longmapsto \Lambda^1$$

by

$$\begin{aligned}
\underline{\mathscr{C}}(X,Y) &:= {}^{c}\underline{\nabla}_X Y - \underline{\nabla}_X Y \\
\underline{\tilde{\mathscr{C}}}(X,Y) &:= {}^{c}\underline{\tilde{\nabla}}_X Y - \underline{\tilde{\nabla}}_X Y \\
\underline{\tilde{\tilde{\mathscr{C}}}}(X,Y) &:= {}^{c}\underline{\tilde{\tilde{\nabla}}}_X Y - \underline{\tilde{\tilde{\nabla}}}_X Y
\end{aligned}$$

where $X, Y \in L^1$ *are vectorfields on* $\underline{\mathscr{M}}$.

The conformal tensor, corresponding to the Vidal and the adapted connection, can most easily be expressed in terms of the conformal tensor of the Levi–Civita connection.

Proposition 4.29
Let $\underline{\mathscr{C}}, \underline{\tilde{\mathscr{C}}}, \underline{\tilde{\tilde{\mathscr{C}}}}$ *be the conformal tensors defined in 4.28 then following relations hold*

$$\begin{aligned}
\underline{\mathscr{C}}(X,Y) &= X[\phi]Y + Y[\phi]X - \underline{g}(X,Y)^{\sharp}\underline{d}\phi \\
\underline{\tilde{\mathscr{C}}}(X,Y) &= \mathcal{P}\underline{\mathscr{C}}(X,\mathcal{P}Y) + \mathcal{P}'\underline{\mathscr{C}}(X,\mathcal{P}'Y) \\
\underline{\tilde{\tilde{\mathscr{C}}}}(X,Y) &= \mathcal{P}\underline{\mathscr{C}}(\mathcal{P}X,\mathcal{P}Y) + \mathcal{P}'\underline{\mathscr{C}}(\mathcal{P}'X,\mathcal{P}'Y)
\end{aligned}$$

where $X, Y \in \Lambda^1$ *are vectorfields on* $\underline{\mathscr{M}}$.

proof: To be added. ∎

The difference between these conformal tensors manifests itself in a clearer way by studying the expressions in component form. The conformal tensor of the Levi–Civita connection is read of from the above proposition and is noticeably symmetric.

Proposition 4.30
Let $\underline{\mathscr{C}}$ *be the conformal tensor of the Levi–Civita connection then in component*

form it reads,

$$
\begin{aligned}
\underline{\mathscr{C}}_{ab}{}^{c} =& 2\delta^{c}_{(a}E_{b)}[\phi] - \eta_{ab}\eta^{cd}E_{d}[\phi] \\
\underline{\mathscr{C}}_{ab}{}^{c'} =& -\eta_{ab}\eta^{c'd'}E_{d'}[\phi] \\
\underline{\mathscr{C}}_{a'b}{}^{c} =& E_{a'}[\phi]\delta^{c}_{b} \\
\underline{\mathscr{C}}_{ab'}{}^{c'} =& E_{a}[\phi]\delta^{c'}_{b'}, \\
\underline{\mathscr{C}}_{a'b'}{}^{c} =& -\eta_{a'b'}\eta^{cd}E_{d}[\phi] \\
\underline{\mathscr{C}}_{a'b'}{}^{c'} =& 2\delta^{c'}_{(a'}E_{b')}[\phi] - \eta_{a'b'}\eta^{c'd'}E_{d'}[\phi]
\end{aligned}
$$

The conformal tensor of the adapted connection can now be derived from the above expressions. It should be stressed though that it is not symmetric.

Proposition 4.31
Let $\underline{\tilde{\mathscr{C}}}$ be the conformal tensor of the adapted connection then in component form it reads

$$
\begin{aligned}
\underline{\tilde{\mathscr{C}}}_{ab}{}^{c} =& 2\delta^{c}_{(a}E_{b)}[\phi] - \eta_{ab}\eta^{cd}E_{d}[\phi] \\
\underline{\tilde{\mathscr{C}}}_{ab}{}^{c'} =& 0 \\
\underline{\tilde{\mathscr{C}}}_{ab'}{}^{c} =& 0 \\
\underline{\tilde{\mathscr{C}}}_{a'b}{}^{c} =& E_{a'}[\phi]\delta^{c}_{b} \\
\underline{\tilde{\mathscr{C}}}_{ab'}{}^{c'} =& E_{a}[\phi]\delta^{c'}_{b'} \\
\underline{\tilde{\mathscr{C}}}_{a'b}{}^{c'} =& 0 \\
\underline{\tilde{\mathscr{C}}}_{a'b'}{}^{c} =& 0 \\
\underline{\tilde{\mathscr{C}}}_{a'b'}{}^{c'} =& 2\delta^{c'}_{(a'}E_{b')}[\phi] - \eta_{a'b'}\eta^{c'd'}E_{d'}[\phi]
\end{aligned}
$$

Finally in the case of the Vidal connection the conformal tensor takes a very simple form and will, like in the Levi–Civita case, be symmetric.

Proposition 4.32
Let $\underline{\tilde{\mathscr{C}}}$ be the conformal tensor of the Vidal connection then in component form

it reads

$$\begin{aligned}
\underline{\tilde{\tilde{\mathscr{C}}}}_{ab}{}^{c} &= 2\delta^{c}_{(a}E_{b)}[\phi] - \eta_{ab}\eta^{cd}E_{d}[\phi] \\
\underline{\tilde{\tilde{\mathscr{C}}}}_{ab}{}^{c'} &= 0 \\
\underline{\tilde{\tilde{\mathscr{C}}}}_{ab'}{}^{c} &= 0 \\
\underline{\tilde{\tilde{\mathscr{C}}}}_{a'b}{}^{c} &= 0 \\
\underline{\tilde{\tilde{\mathscr{C}}}}_{ab'}{}^{c'} &= 0 \\
\underline{\tilde{\tilde{\mathscr{C}}}}_{a'b}{}^{c'} &= 0 \\
\underline{\tilde{\tilde{\mathscr{C}}}}_{a'b'}{}^{c} &= 0 \\
\underline{\tilde{\tilde{\mathscr{C}}}}_{a'b'}{}^{c'} &= 2\delta^{c'}_{(a'}E_{b')}[\phi] - \eta_{a'b'}\eta^{c'd'}E_{d'}[\phi]
\end{aligned}$$

Well known is the fact that when decomposing the Riemann curvature tensor of the Levi–Civita connection into its irreducible parts with respect to its traces, the appearing Weyl tensors measures whether the riemannian manifold is conformally flat or not. More specifically, the vanishing of the Weyl tensor is the condition for local conformal flatness in the case when the dimension of the manifold exceeds three. The tracefree part of the Ricci tensor is defined by,

$$\underline{\hat{R}}_{\bar{a}\bar{b}} := \underline{R}_{\bar{a}\bar{b}} - \frac{1}{m}\underline{R}\eta_{\bar{a}\bar{b}} \tag{16}$$

Next the Ricci one-form and the tracefree Ricci one-form are defined.

Definition 4.33
Let $\underline{R}_{ab}, \underline{\hat{R}}_{ab}$ denote the Ricci tensor and its tracefree part respectively then define their one forms by

$$\begin{aligned}
\underline{R}^{\bar{a}} &:= E^{\bar{b}}\underline{R}_{\bar{b}}{}^{\bar{a}} \\
\underline{\hat{R}}^{\bar{a}} &:= E^{\bar{b}}\underline{\hat{R}}_{\bar{b}}{}^{\bar{a}} = \underline{R}^{\bar{a}} - \frac{1}{m}\underline{R}E^{\bar{a}}
\end{aligned}$$

Denoting the Rimann two form $\underline{R}^{\underline{cd}} = \frac{1}{2}E^{\underline{a}} \wedge E^{\underline{b}}\underline{R}_{\underline{ab}}{}^{\underline{cd}}$ and the Weyl two form by $\underline{C}^{\underline{cd}} = \frac{1}{2}E^{\underline{a}} \wedge E^{\underline{b}}\underline{C}_{\underline{ab}}{}^{\underline{cd}}$ the decomposition is most elegantly written

$$\begin{array}{ccccccc}
\frac{1}{12}m^2(m^2-1) & = & \frac{1}{12}(m-3)m(m+1)(m+2) & + & \frac{1}{2}(m-1)(m+2) & + & 1 \\
\underline{R}^{\bar{c}\bar{d}} & = & \underline{C}^{\bar{c}\bar{d}} & + & \frac{2}{m-2}\underline{\hat{R}}^{[\bar{c}} \wedge E^{\bar{d}]} & + & \frac{1}{m(m-1)}\underline{R}E^{\bar{c}} \wedge E^{\bar{d}}
\end{array}$$

After reinserting the expression for the tracefree part of the Ricci tensor and solving for the Weyl tensor the more familiar form is obtained

$$\underline{C}_{\bar{a}\bar{b}}{}^{\bar{c}\bar{d}} = \underline{R}_{\bar{a}\bar{b}}{}^{\bar{c}\bar{d}} - \frac{4}{m-2}\delta^{[\bar{c}}_{[\bar{a}}\underline{R}^{\bar{d}]}_{\bar{b}]} + \frac{2}{(m-1)(m-2)}\underline{R}\delta^{[\bar{c}}_{[\bar{a}}\delta^{\bar{d}]}_{\bar{b}]} \tag{17}$$

From this equation the different components are read off and put in next proposition.

Proposition 4.34
Let $\underline{C}^{\bar{c}\bar{d}}$ be the weyl tensor in an almost product manifold then its components look like

$$\begin{aligned}
\underline{C}_{ab}{}^{cd} =& \underline{R}_{ab}{}^{cd} - \frac{4}{m-2}\delta^{[c}_{[a}\underline{R}^{d]}_{b]} + \frac{2}{(m-1)(m-2)}\underline{R}\delta^{[c}_{[a}\delta^{d]}_{b]}\\
\underline{C}_{ab}{}^{cd'} =& \underline{R}_{ab}{}^{cd'} - \frac{2}{m-2}\delta^{c}_{[a}\underline{R}^{d'}_{b]}\\
\underline{C}_{ab}{}^{c'd'} =& \underline{R}_{ab}{}^{c'd'}\\
\underline{C}_{a'b}{}^{c'd} =& \underline{R}_{a'b}{}^{c'd} - \frac{1}{m-2}(\delta^{c'}_{a'}\underline{R}^{d}_{b} + \delta^{d}_{b}\underline{R}^{c'}_{a'}) + \frac{1}{(m-1)(m-2)}\underline{R}\delta^{c'}_{a'}\delta^{d}_{b}\\
\underline{C}_{a'b'}{}^{c'd} =& \underline{R}_{a'b'}{}^{c'd} - \frac{2}{m-2}\delta^{c'}_{[a'}\underline{R}^{d}_{b']}\\
\underline{C}_{a'b'}{}^{c'd'} =& \underline{R}_{a'b'}{}^{c'd'} - \frac{4}{m-2}\delta^{[c'}_{[a'}\underline{R}^{d']}_{b']} + \frac{2}{(m-1)(m-2)}\underline{R}\delta^{[c'}_{[a'}\delta^{d']}_{b']}
\end{aligned}$$

From above proposition it is for instance clear that the $\underline{R}_{abc'}{}^{d'}$ component of the Riemann tensor must be conformally invariant. Taking a look at its final expression in proposition 4.24 the only non-manifest conformally invariant terms, are those involving derivatives. These can be proved to be independently conformally invariant.

Proposition 4.35
Let L, L' be the respective twisting tensors of the characteristic distributions defined by an almost product structure on $\underline{\mathscr{M}}$ then the following relations hold

$$\begin{aligned}
\hat{\underline{\tilde{\nabla}}}_{[Z'}L_{W']}(X,Y) =& e^{2\phi}\underline{\tilde{\nabla}}_{[Z'}L_{W']}(X,Y)\\
\hat{\underline{\tilde{\nabla}}}_{[Z}L'_{W]}(X',Y') =& e^{2\phi}\underline{\tilde{\nabla}}_{[Z}L'_{W]}(X',Y')
\end{aligned}$$

or put in component form

$$\begin{aligned}
(\hat{\underline{\tilde{\nabla}}}_{[c'}L)_{ab|d']} =& e^{2\phi}(\underline{\tilde{\nabla}}_{[c'}L)_{ab|d']}\\
(\hat{\underline{\tilde{\nabla}}}_{[c}L')_{a'b'|d]} =& e^{2\phi}(\underline{\tilde{\nabla}}_{[c}L')_{a'b'|d]}
\end{aligned}$$

Proceeding in the same fashion as in [8],investigating how the curvature components of the adapted connection can be divided into irreducible parts - in the case of an almost product manifold instead as for just an embedding, some immediate differences is noticed. The generalisation of what Carter calls the outer curvature is the $\underline{\tilde{R}}_{ab}{}^{c'd'}$ component which will be seen not to be conformally invariant in the generic case but only if the unprimed distribution is integrable. The same is of course true for the outer curvature of the complementary (primed) distribution. From proposition 4.19 it is manifest that the non-invariant components are

$$\frac{2}{k} L_{ab[c'}\kappa_{d']}$$

and

$$\frac{2}{k'} L'_{a'b'[c}\kappa'_{d]}$$

respectively. For the internal curvature which in the language of almost product manifolds is the total semi-basic components of the adapted curvature, one can follow the procedure of dividing the tensor components into its irreducible parts according to the above scheme. In the generic case the semi-basic components does not have the box symmetry

$$\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}$$

. Because of torsion though, it has the other symmetry parts. It is clear that when the distribution is integrable its internal curvature will indeed have the box symmetry. Defining the internal Weyl tensors the generalization of [8, 9] can be made.

Definition 4.36
Let $\underline{\tilde{R}}$ be the curvature tensor of the adapted connection and $\underline{\tilde{R}}_{ab}{}^{cd}$, $\underline{\tilde{R}}_{a'b'}{}^{c'd'}$ the internal curvatures of the two complementary distributions associated with an almost product manifold then define their respective Weyl tensors by

$$\begin{aligned}
\tilde{C}_{ab}{}^{cd} &= \tilde{R}_{ab}{}^{cd} - \frac{4}{k-2}\delta^{[c}_{[a}\tilde{R}^{d]}_{b]} + \frac{2}{(k-1)(k-2)}\tilde{R}\delta^{[c}_{[a}\delta^{d]}_{b]} \\
\tilde{C}'_{a'b'}{}^{c'd'} &= \tilde{R}'_{a'b'}{}^{c'd'} - \frac{4}{k'-2}\delta^{[c'}_{[a'}\tilde{R}'^{d']}_{b']} + \frac{2}{(k'-1)(k'-2)}\tilde{R}'\delta^{[c'}_{[a'}\delta^{d']}_{b']}
\end{aligned}$$

where

$$\begin{aligned}
\tilde{R}_{ab}{}^{cd} &:= \underline{\tilde{R}}_{ab}{}^{cd}, & \tilde{R}^{b}_{a} &:= \underline{\tilde{R}}_{ac}{}^{bc}, & \tilde{R} &:= \underline{\tilde{R}}_{ab}{}^{ab}, \\
\tilde{R}'{}_{a'b'}{}^{c'd'} &:= \underline{\tilde{R}}_{a'b'}{}^{c'd'}, & \tilde{R}'^{b'}_{a'} &:= \underline{\tilde{R}}_{a'c'}{}^{b'c'}, & \tilde{R}' &:= \underline{\tilde{R}}_{a'b'}{}^{a'b'}.
\end{aligned}$$

Following Carter's procedure it is now easy to generalize his relations to the case of an almost product manifold.

Proposition 4.37
Let $\tilde{C}, \tilde{C}'$ be the Weyl tensors of the internal curvatures of the two complemenatry distributions associated with an almost product manifold. Then the

following relations hold.

$$
\begin{aligned}
\tilde{C}_{ab}{}^{cd} =& C_{ab}{}^{cd} - \frac{4}{k-2}\delta^{[c}_{[a}C^{d]}_{b]} + \frac{2}{(k-1)(k-2)}C - \\
& - 2(L_{[a|}{}^{c}{}_{e'}L_{b]}{}^{de'} + 2L_{[a|}{}^{[c}{}_{e'}W_{b]}{}^{d]e'}) - \\
& - \frac{4}{k-2}(\delta^{[c}_{[a|}L_{b]}{}^{e}{}_{e'}L_{e}{}^{d]e'} + \delta^{[c}_{[a|}W_{b]}{}^{e}{}_{e'}W_{e}{}^{d]e'} + \delta^{[c}_{[a|}L_{b]}{}^{e}{}_{e'}W_{e}{}^{d]e'} + \\
& + \delta^{[c}_{[a|}W_{b]}{}^{e}{}_{e'}L_{e}{}^{d]e'}) + \\
& + \frac{2}{(k-1)(k-2)}\delta^{c}_{[a}\delta^{d}_{b]}(W^2 - L^2), \\
\tilde{C}'_{a'b'}{}^{c'd'} =& C'_{a'b'}{}^{c'd'} - \frac{4}{k'-2}\delta^{[c'}_{[a'}C'^{d']}_{b']} + \frac{2}{(k'-1)(k'-2)}C' - \\
& - 2(L'_{[a'|}{}^{c'}{}_{e}L'_{b']}{}^{d'e} + 2L'_{[a'|}{}^{[c'}{}_{e}W'_{b']}{}^{d']e}) - \\
& - \frac{4}{k'-2}(\delta^{[c'}_{[a'|}L'_{b']}{}^{e'}{}_{e}L'_{e'}{}^{d']e} + \delta^{[c'}_{[a'|}W'_{b']}{}^{e'}{}_{e}W'_{e'}{}^{d']e} + \delta^{[c'}_{[a'|}L'_{b']}{}^{e'}{}_{e}W'_{e'}{}^{d']e} + \\
& + \delta^{[c'}_{[a'|}W'_{b']}{}^{e'}{}_{e}L'_{e'}{}^{d']e}) + \\
& + \frac{2}{(k'-1)(k'-2)}\delta^{c'}_{[a'}\delta^{d'}_{b']}(W'^2 - L'^2)
\end{aligned}
$$

where

$$
\begin{aligned}
C_{ab}{}^{cd} :=& \underline{C}_{ab}{}^{cd}, & C^{b}_{a} :=& C_{ac}{}^{bc}, & C :=& C_{ab}{}^{ab} \\
C'{}_{a'b'}{}^{c'd'} :=& \underline{C}_{a'b'}{}^{c'd'}, & C'^{b'}_{a'} :=& C'_{a'c'}{}^{b'c'}, & C' :=& C'_{a'b'}{}^{a'b'}.
\end{aligned}
$$

From this proposition it follows that if the semi-basic part of the conformal tensor is zero and the distribution is integrable then the distribution possess local conformal flatness if and only if its conformation tensor vanishes. Of course this is only true in the case where $k > 3$. This generalizes Carter's result to the case of almost product manifold.

5 Physical applications

There are lots of physical applications involving almost product manifolds. Because principal bundles can be regarded as a almost product manifold with the (GF, GD) structure ordinary gauge theory can be found in its utmost geometrical form. In Kaluza–Klein theory the internal space need not be a group manifold but could instead be a homogenous space with the proper gauge group as its isometry group. Here will be given an example of the recovered Kaluza–Klein theory from the almost product structure taken first in the most general case where no restriction of the fiber is made.

Example 5.1

Kaluza-Klein theory

Here will be seen how, in the case of a (GF, GD) structure, the Vidal connection will reduce to the gauge covariant derivative. The Einstein–Hilbert action will reduce to the inner curvatures plus the gauge field term as in [10]. First note the spliting of the action in the (GF, GD) case,

$$\int d^m x\sqrt{\underline{g}}\underline{R} = \int d^k x d^{k'} y\sqrt{g}\sqrt{g'}(\tilde{\tilde{R}} + \tilde{\tilde{R}}' + L^2). \tag{18}$$

Note that the primed distribution is chosen to be integrable. Following [10] the vielbeins can locally be parametrized as

$$\begin{aligned} \underline{E}_a &= E_a + A_a^i K_i, & \underline{E}_{a'} &= E_{a'} \\ \underline{E}^a &= E^a, & \underline{E}^{a'} &= E^{a'} - E^a A_a^i K_i^{a'} \end{aligned}$$

where $K_i = K_i{}^{a'}(y)E_{a'}(y)$ are the Killing vectors of the integrable internal manifold. These satisfy an algebra

$$[K_i, K_j] = f_{ij}{}^k K_k. \tag{19}$$

where the structure constants $f_{ij}{}^k$ of the isometry group and does not depend on y. The gauge fieldstrength $F^i = dA^i + \frac{1}{2}f_{jk}{}^i A^j \wedge A^k$ is most easily found [1]

$$F(X, Y) = \mathcal{P}'[X, Y] = 2L(X, Y) \tag{20}$$

where X, Y are unprimed vector fields which implies that the L^2 term in the action reads $\frac{1}{4}F^2$ which is the ordinary action term in gauge theory. Now the Vidal connection of the gauge field can be written

$$(\tilde{\tilde{\underline{\nabla}}}_X F)(Y, Z) = (\underline{\nabla}_X^{\mathfrak{D}} F)(Y, Z) + F^i(Y, Z)\mathcal{P}'[X, K_i], \tag{21}$$

written in component form, the relations look like,

$$(\tilde{\tilde{\underline{\nabla}}}_a F)^i_{bc} = (\underline{\nabla}_a^{\mathfrak{D}} F)^i_{bc} + f^i_{jk} A^j_a F^k_{bc} \tag{22}$$

which is precisely the gauge covariant derivative. Further the identity from proposition 4.10 $(\tilde{\tilde{\underline{\nabla}}}_{[a} L)_{bc]}{}^{d'} = 0$ reduces to the Bianchi identity of the gauge field

$$(\tilde{\tilde{\underline{\nabla}}}_{[a} F)^i_{bc]} = 0 \tag{23}$$

the $\underline{R}_{ab'}$ term from proposition 4.27 reduces to

$$\underline{R}_{ab'} = \frac{1}{2}(\tilde{\tilde{\underline{\nabla}}}_c F)_a^{i\,c} K_{ib'} \tag{24}$$

which from the Einstein's equations point of view reduces to the equations of motion for the gauge field, i.e. $(\tilde{\tilde{\underline{\nabla}}}_c F)_a^{i\,c} = 0$. So it is clear that gauge theory and Kaluza-Klein theory is contained in the almost product manifold description. In the general case however the Killing vectors could be exchanged to ordinary vielbeins, the structure coefficients need not be constant and the fiber no longer a group space or homogenous space, the almost product structure procedure would still be valid. In the case of Kaluza-Klein theory containing the dilaton field it is easy to see that it is contained in the mean curvature, κ, see [1].

Another more traditional example is the decomposition of the four dimensional curvature scalar into three space in hamiltonian formulation of ordinary Einstein gravity. From proposition 4.26 the decomposition of the curvature scalar is immediately found to be $\underline{R} = \underline{\tilde{R}} - \kappa^2 + K^2$. From a foliation point of view it is clear that a space with non-degenerate metric of Minkowskian signature must have vanishing euler number which is also the condition for a space to have a codimension one foliation [11, 12]. This is why the L^2 term can be set to zero.

6 Conclusions and outlook

The theory of almost product manifolds is seen to overlap with a lot of physical applications. The main areas is of course geometrical phyics such as gravity, Kaluza-Klein theory and ordinary gauge theory. Here was seen for instance that in gauge- or Kaluza-Klein theory the Vidal connection reduces to the ordinary gauge covariant derivative and the second curvature identity of the Vidal curvature gives the Bianchi identity of the gauge field. The relations found propositions 4.26 and 4.27 could perhaps be used to find new solutions to the equations of motions of various supergravity theories. Here the the Ricci tensors are given in the most general case why all kind of brane solutions must fit in this scheme. From these relations it should also be clear that a black hole solution in ordinary space-time carrying a gauge field charge should correspond to a black hole solution in the total space with rotational parameters corresponding to the gauge charges. The reason for this is of course the identification of the twisting tensor as the gauge fieldstrength. In super gravity theories the black hole solutions carry charges from anti-symmetric tensor fields and will be p-brane solutions. These can again have rotational parameters in the transverse directions which correspond to gauge field charges of the isometry group of the transverse space [3, 13] which by analysis in [1] would correspond to a non-integrability of the brane itself in this context.

Another interesting investigation would be to see how the Clifford algebra splits in a almost product manifold. In a appearing paper [14] will be shown how flat super space looks in the almost product structure picture.

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A Tic–Tac–Toe notation

The Tic-Tac-Toe notation can most easily be described by working through the first non-trivial Young tableau, which is the (2,1) one.

$$\begin{ytableau} \ & \ \\ \ \end{ytableau} = \begin{ytableau} o & o \\ o \end{ytableau} \oplus \begin{ytableau} o & x \\ o \end{ytableau} \oplus \begin{ytableau} o & o \\ x \end{ytableau} \oplus \begin{ytableau} x & x \\ o \end{ytableau} \oplus \begin{ytableau} x & o \\ x \end{ytableau} \oplus \begin{ytableau} x & x \\ x \end{ytableau} \tag{25}$$

Above the Tic-Tac-Toe notation was used where the o's labels unprimed degrees of freedom and the x's primed ones. The Tic-Tac-Toe tableaux works exactly as an ordinary Young-tableau. Since the primed and the unprimed directions does not talk with each other when it comes to symmetries, the dimension of a Tic-Tac-Toe tableau equals the product of the dimensions of the pure primed and unprimed sub-tableaux respectively. Their respective dimensions read

$$\frac{m(m^2-1)}{3} = \frac{k(k^2-1)}{3} + \frac{k(k-1)k'}{2} + \frac{k(k+1)k'}{2} + \\ + \frac{k'(k'+1)k}{2} + \frac{k'(k'-1)k}{2} + \frac{k'(k'^2-1)}{3}$$

where $k' = m - k$. The decomposition of an arbitrary Young tableau can be done in a similar fashion.

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